Part IA Engineering Mathematics: Lent Term

Convolution

Fourier Series

Probability

Joan Lasenby
Lent 2014

Contents and Examples Questions

****

Contents and Examples Questions cont...

Section 1

Linear Systems & Impulse Response

- We motivate the study of **linear time-invariant systems**.
- The principle of superposition is explained.
- Step functions and delta functions are introduced, together with their corresponding responses.
- Examples are given to illustrate the use of the step response with superposition.
- The **sifting theorem** is stated and illustrated with some examples.

Motivation

Many engineering problems concern linear systems.

Forces → System → Strains

Voltages → System → Currents

Pressure → System → Density

Heat flow → System → Temperature

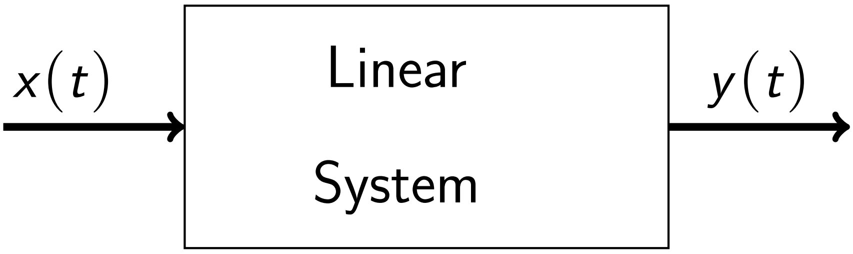
Power → System → Kinetic Energy

In a linear system the output ($y(t)$) is computed as some linear combination of the input ($x(t)$) (including inputs from the past, if we are considering a system with a time-varying input and output).

$$y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t) = F(\alpha_1 x_1(t) + \alpha_2 x_2(t))$$

where $y_i(t) = F(x_i(t))$.

Linear Systems



We will first of all consider Linear *Time Invariant* Systems – define these:

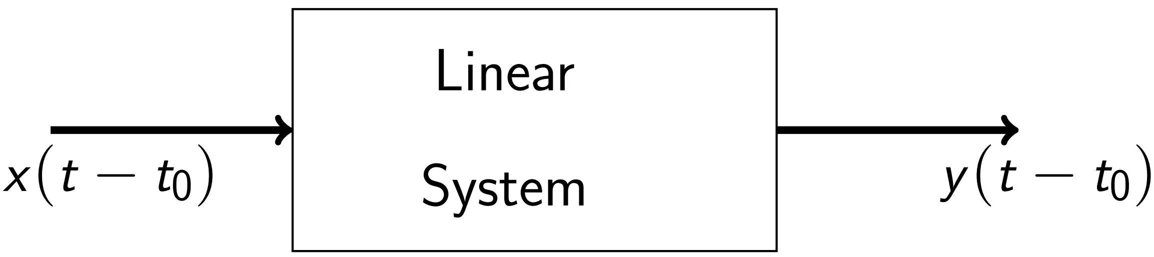
1. Linear time-invariant (LTI) systems satisfy the principle of superposition.

   $$\begin{array}{llll} \text{If input } x_1(t) & \to & \text{output} & y_1(t) \\ \text{and input } x_2(t) & \to & \text{output} & y_2(t) \\ & \text{then} & & \\ \text{input } \alpha x_1(t) + \beta x_2(t) & \to & \text{output} & \alpha y_1(t) + \beta y_2(t) \end{array}$$

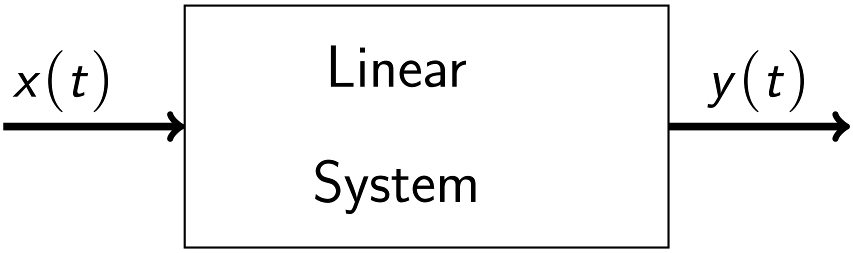
   where α and β are any constants.

2. LTI systems have the special property that if we shift the input we shift the output, ie

$$x(t - t_0) \to y(t - t_0)$$



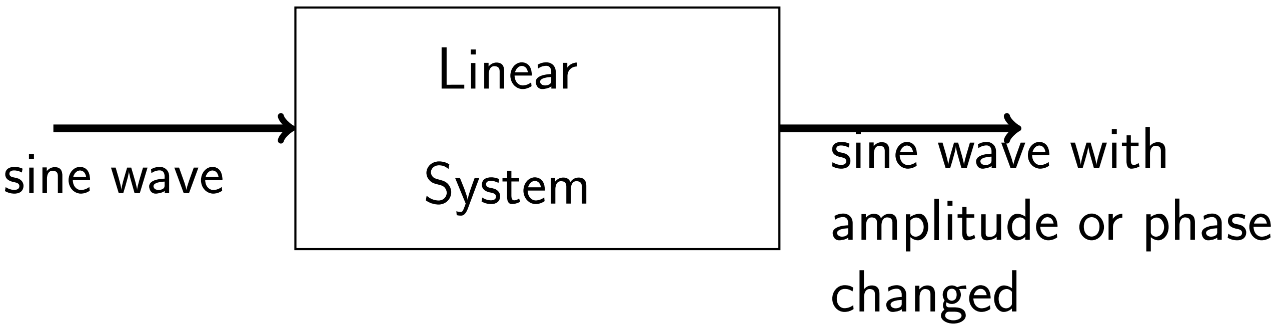
Linear Systems – Again!



1. Linear time-invariant (LTI) systems satisfy the principle of superposition.

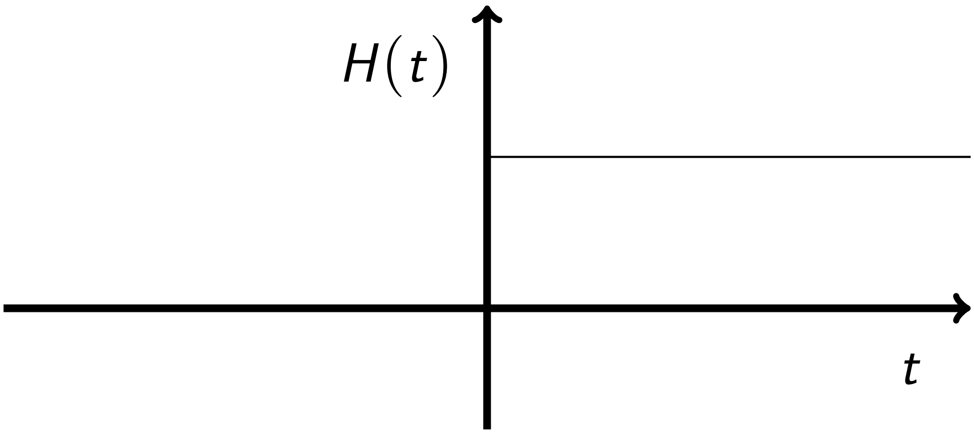
   If input $x_1(t)$ $\rightarrow$ output $y_1(t)$
   and input $x_2(t)$ $\rightarrow$ output $y_2(t)$
   then
   input $\alpha x_1(t) + \beta x_2(t)$ $\rightarrow$ output $\alpha y_1(t) + \beta y_2(t)$
   where α and β are any constants.

2. LTI systems have the special property that a sine wave at the input leads to a sine wave of the same frequency at the output, but with possible changes in amplitude and phase.

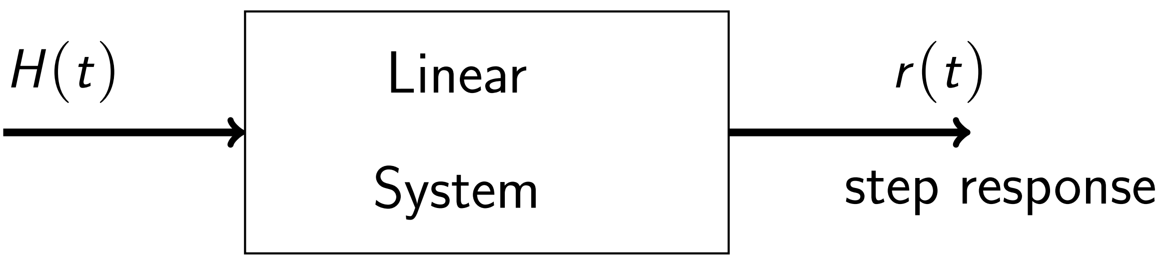


These 2 definitions of LTI systems are equivalent.

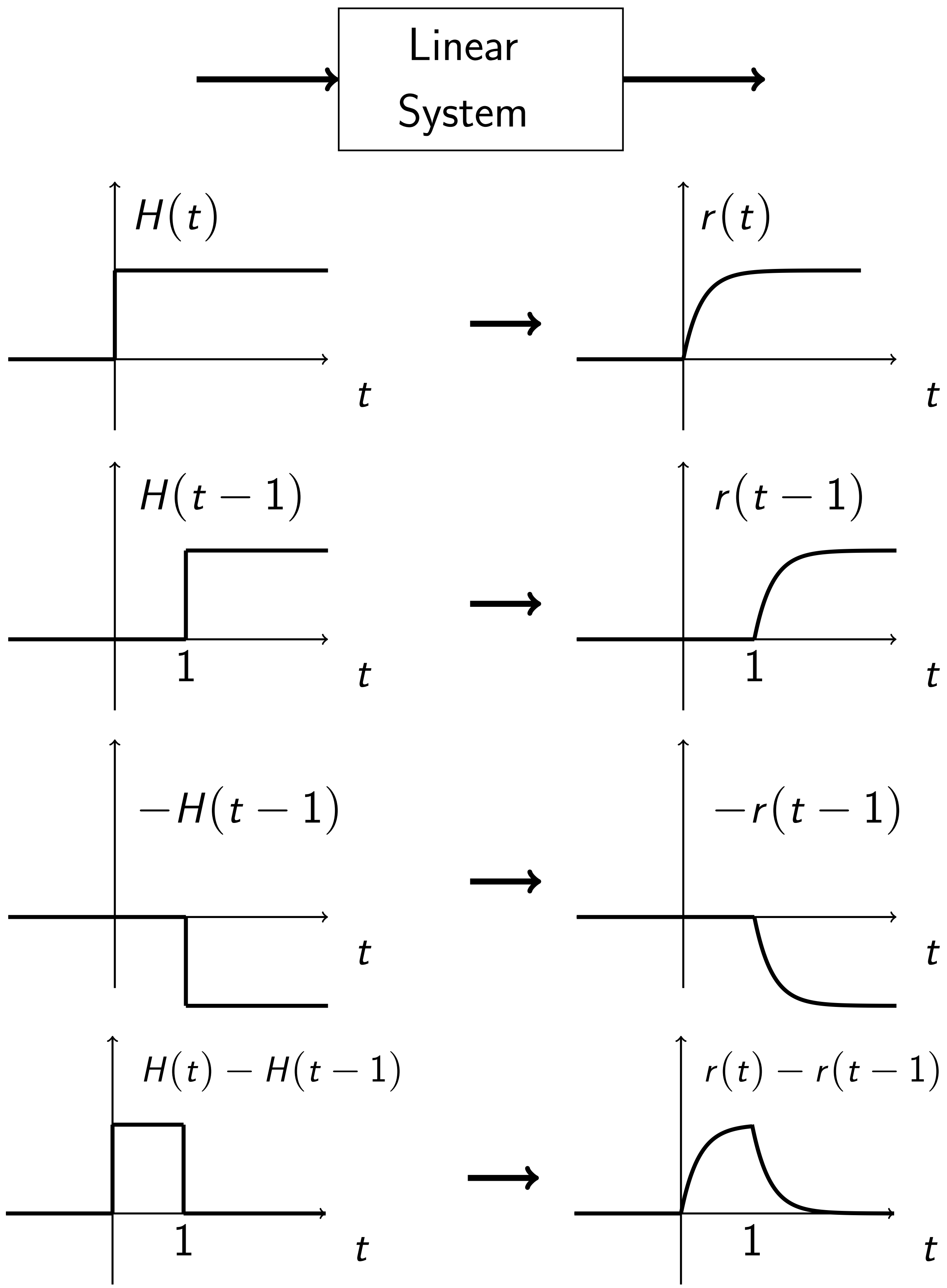
Step Function



$$H(t) = \begin{cases} 0 & t < 0 \\ 1, & t > 0 \end{cases}$$



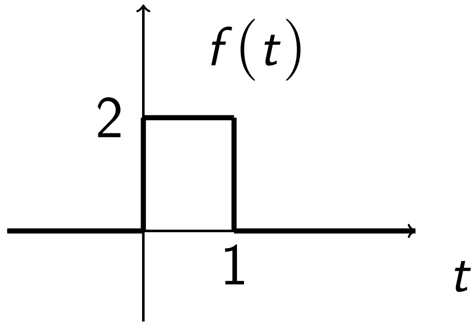
Superposition Example



Calculation of Superposition

Find the output of a linear system with step response:

$$r(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-5t} & t \geq 0 \end{cases}$$

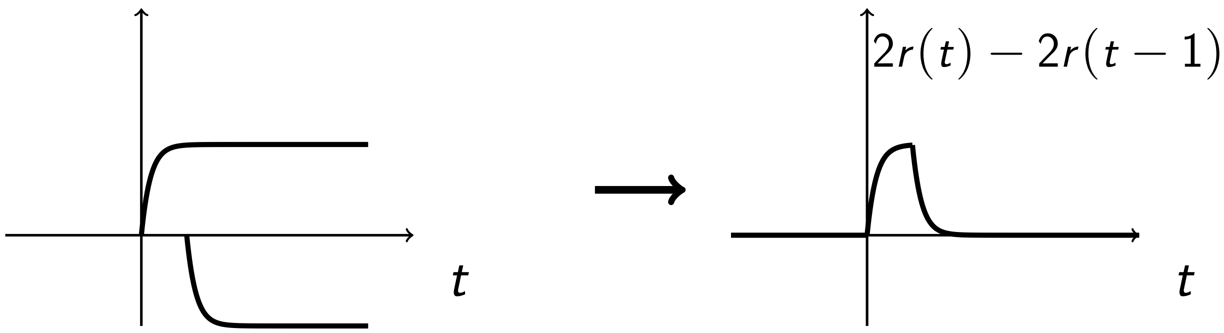


when the input is the pulse $f(t)$.

From our principle of superposition we know that since our pulse can be written as: $f(t) = 2H(t) - 2H(t-1)$ then our output $y(t)$ is given by

$$y(t) = 2r(t) - 2r(t-1)$$

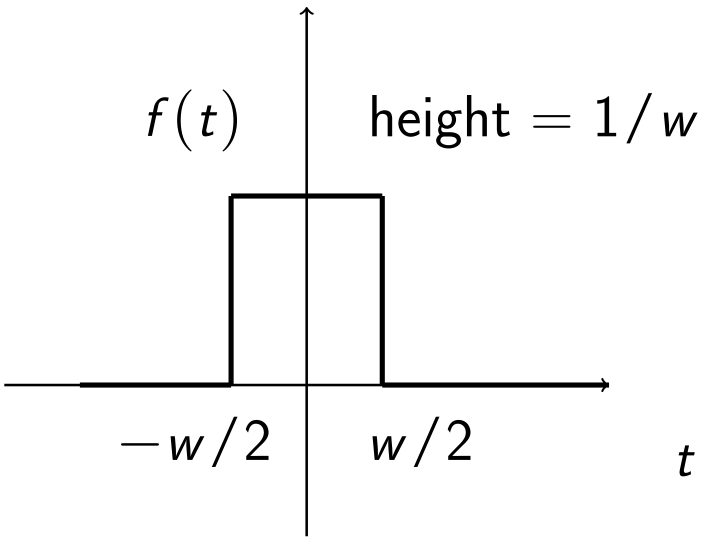
which we visualise as:



Thus, since, for $t \geq 1$,
$2r(t) - 2r(t-1) = 2\left[1 - \mathrm{e}^{-5t} - 1 + \mathrm{e}^{-5(t-1)}\right]$

$$y(t) = \begin{cases} 2(1 - \mathrm{e}^{-5t}) & 0 \leq t < 1 \\ 2\left(\mathrm{e}^{5} - 1\right)\mathrm{e}^{-5t} & t \geq 1 \end{cases}$$

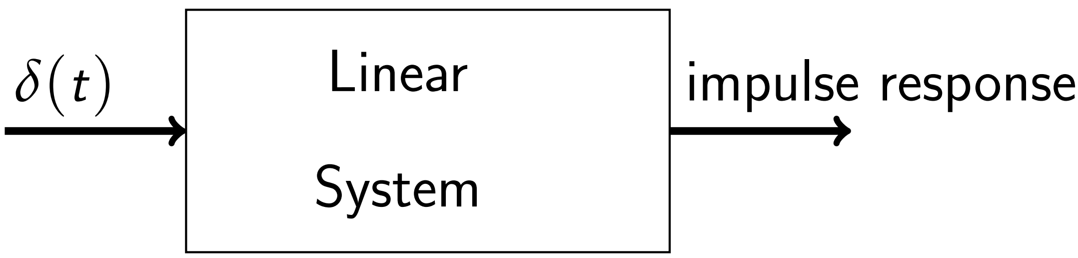
The Dirac Delta Function



As $w \to 0$ the pulse $f(t)$ becomes narrower and taller. In the limit as $w \to 0$ the pulse $f(t)$ becomes a delta function: $\delta(t)$.

The delta function is a spike with unit area. It tends to infinity when its argument tends to zero.

$$
\begin{aligned}
\delta(t) &= 0 \quad \text{except at } t = 0 \\
\int_a^b \delta(t)\,dt &= 1 \quad \text{provided } a < 0 \text{ and } b > 0
\end{aligned}
$$



Integrating the Delta Function

From the previous page

$$\int_a^b \delta(t)\,dt = 1 \quad \text{provided } a < 0 \text{ and } b > 0$$

thus

$$\begin{aligned} \int_{-\infty}^{T} \delta(t)\,dt &= \begin{cases} 0 & T < 0 \\ 1 & T > 0 \end{cases} \\ &= H(T) \end{aligned}$$

Thus, the integral of a delta function is a step function.

Conversely, the derivative of a step function is a delta function.

Delta Function $\delta(t)$ $\xrightarrow{\text{integrate}}$ **Step Function** $H(t)$

Step Function $H(t)$ $\xrightarrow{\text{differentiate}}$ **Delta Function** $\delta(t)$

Impulse Response

From this relation between $\delta(t)$ and the step function $H(t)$, and what we know about LTI systems, we can deduce:

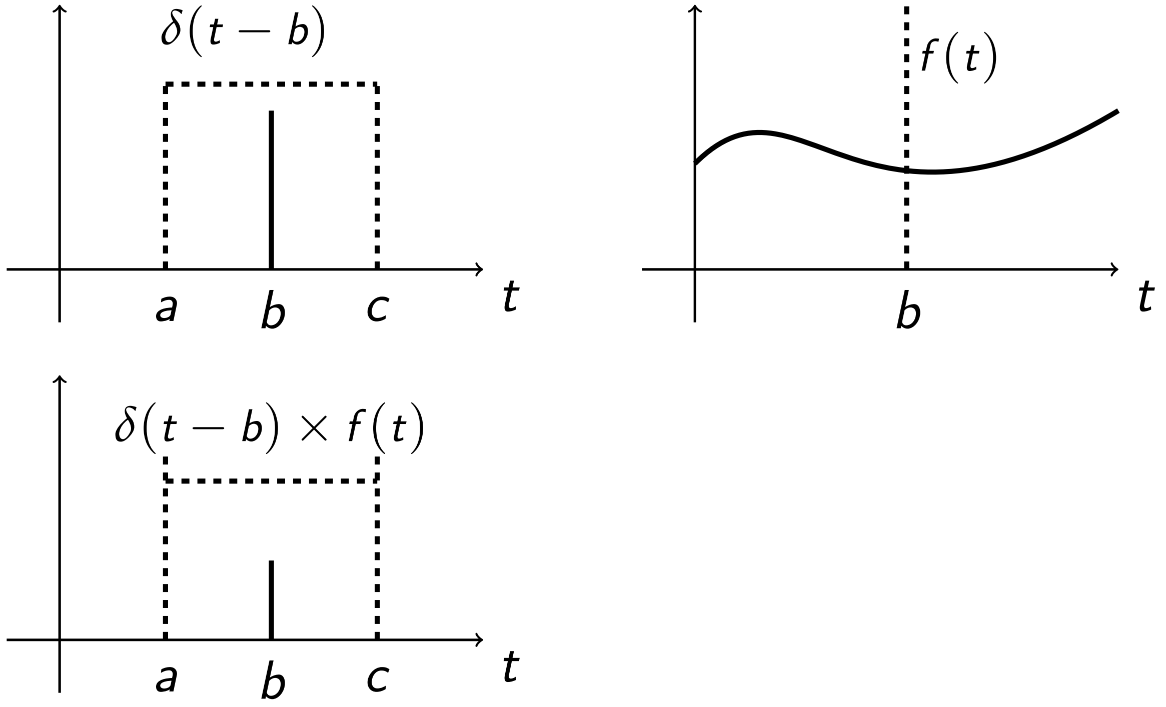
Impulse Response $g(t)$	integrate $\longrightarrow$ differentiate $\longleftarrow$	**Step Response** $r(t)$

Example: Find the output, $g(t)$ of a linear system with step response $r(t) = 1 - e^{-5t}$ when the input is the delta function $\delta(t)$.

$$
\begin{aligned}
r(t) &= 1 - e^{-5t} \\
g(t) &= \frac{dr}{dt} = 5e^{-5t}
\end{aligned}
$$

So the impulse response of the system is $5e^{-5t}$.

Sifting Theorem



$$\int_a^c \delta(t-b)dt = 1 \quad \text{provided } a < b \text{ and } c > b$$

Thus

$$\int_a^c \delta(t-b)f(t)dt = f(b) \quad \text{provided } a < b \text{ and } c > b$$

Sifting Examples

$$\int_{-\pi}^{\pi} \cos(2t)\,\delta(t)\,dt \;=\; \cos(0) = 1$$

$$\int_{-\pi}^{\pi} \cos(2t)\,\delta\left(t-\frac{\pi}{2}\right)\,dt \;=\; \cos(\pi) = -1$$

$$\int_{-\pi}^{0} \cos(2t)\,\delta\left(t-\frac{\pi}{2}\right)\,dt \;=\; 0$$

$$\int_{-\pi}^{\pi} t\,\delta\left(t+\frac{\pi}{2}\right)\,dt \;=\; -\frac{\pi}{2}$$

$$\int_{0}^{\pi} t\,\delta\left(t+\frac{\pi}{2}\right)\,dt \;=\; 0$$

Section 1: Summary

Superposition (for linear systems):

If input $f_1(t)$	$\rightarrow$	output $y_1(t)$
and input $f_2(t)$	$\rightarrow$	output $y_2(t)$ then
input $\alpha f_1(t) + \beta f_2(t)$	$\rightarrow$	output $\alpha y_1(t) + \beta y_2(t)$

where α and β are any constants.

Sifting:

$$\int_a^c \delta(t-b) f(t) dt = f(b) \quad \text{provided } a < b \text{ and } c > b$$

- Step function and step response.
- Impulse function and impulse response.
- Finding the system response to a pulse by combining scaled and delayed step responses using superposition.

Section 2

Differential Equations to describe Linear Systems

We motivate the *convolution integral*, which will be presented in Section 3, using an example of a car going up a step.

A technique is described for solving a linear differential equation to obtain the step response of the system. We set the input to 1, and solve with initial conditions $y = \dot{y} = 0$ for $t = 0$. The impulse response can then be obtained by differentiating the step response.

The utility of this technique, when used together with convolution, is outlined.

Differential Equations

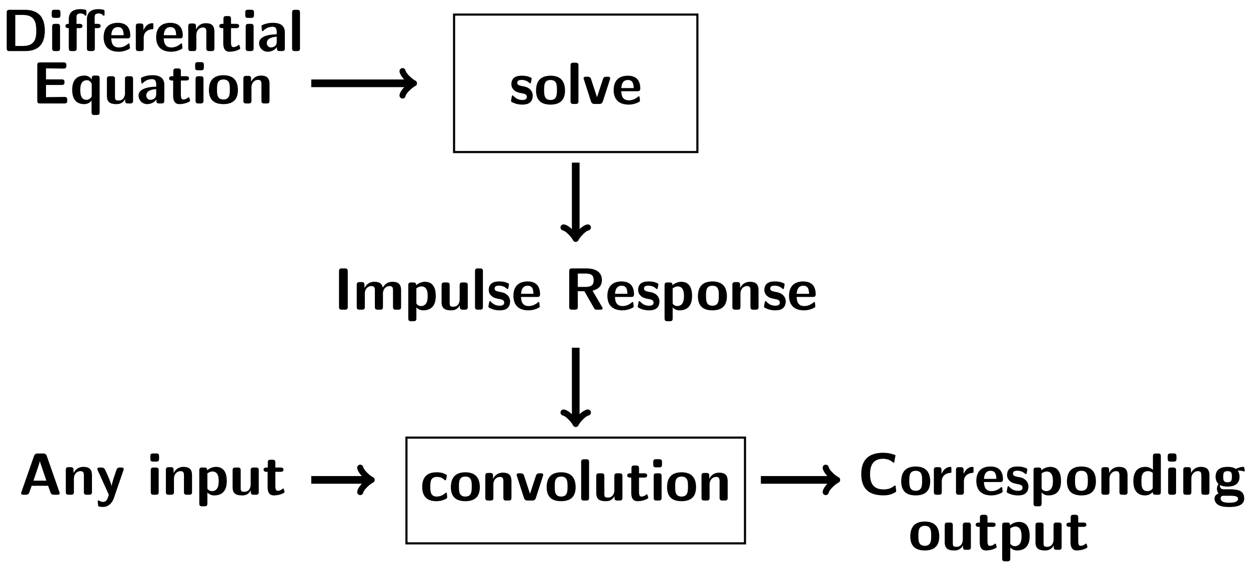
Linear systems are often described using differential equations. For example:

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = f(t)$$

where $f(t)$ is the input to the system and $y(t)$ is the output.

We know how to solve for y given a specific input f.

We now cover an alternative approach:



Solving for Impulse Response

We cannot (given our current knowledge) solve for the impulse response directly so we solve for the step response and then differentiate it to get the impulse response.

Differential Equation $\longrightarrow$ $\boxed{\textbf{solve}}$

$\downarrow$

Step Response

$\downarrow$

$\boxed{\textbf{differentiate}}$

$\downarrow$

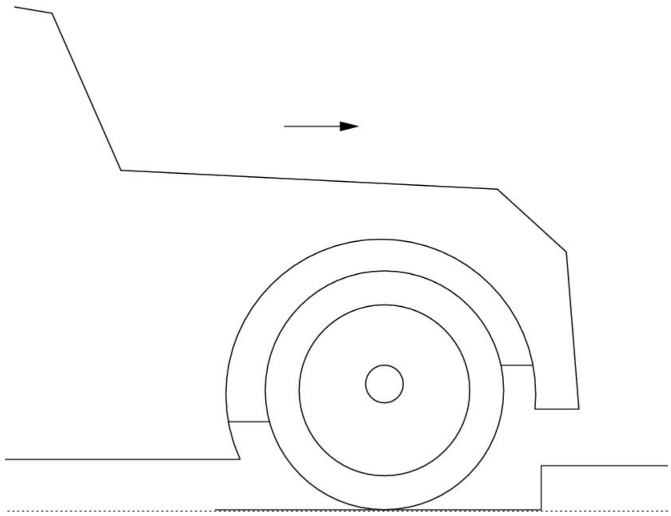
Impulse Response

$\downarrow$

Any input $\longrightarrow$ $\boxed{\textbf{convolution}}$ $\longrightarrow$ **Corresponding output**

Motivation: Convolution

If we know the response of a linear system to a step input, we can calculate the impulse response and hence we can find the response to any input by convolution (this is an assertion at present!).



Suppose we want to know how a car's suspension responds to lots of different types of road surface.

We measure how the suspension responds to a step input (or calculate the step response from a theoretical model of the system).

We can then find the impulse response and use convolution to find the car's behaviour for any road surface profile.

Solving for Step Response

Suppose we want to find the step response of

$$\frac{dy}{dt} + 5y = f(t)$$

where f is the input and y is the output. It would be nice if we could put $f(t) = H(t)$ and solve. Unfortunately we don't know of a way to do this directly. So we

1. set $f(t) = 1$, and solve for just $t \geq 0$
2. set the boundary condition $y(0) = 0$ (also $\dot{y}(0) = 0$ for second order equations). Note then that if we assume we have a causal system in which $y(t) = 0$ for $t < 0$, we require $f(t) = 0$ for $t < 0$.

We thus have a solution to our *step function* input, because $y = 0$ for $t < 0$, and we have found y for all $t \geq 0$.

Boundary Condition Justification

Prove that $y = 0$ at $t = 0$ by contradiction.

We know that $y(t) = 0$ for all $t < 0$. Therefore the only way for y to equal something other than zero at $t = 0$ is if there is a step discontinuity in y at $t = 0$.

Assume that y has a step of height h at $t = 0$. If y has a step discontinuity at $t = 0$ then $\frac{dy}{dt}$ must have a delta function at $t = 0$.

So we have:

- $f(t)$ is a step function so $|f(t)| \leq 1$ for all t.
- $|y| \leq h$ at $t = 0$.
- $\left|\frac{dy}{dt}\right| \to \infty$ at $t = 0$.

Which violates the original equation at $t = 0$.

$$\frac{dy}{dt} = f(t) - 5y$$

As the RHS is finite but the LHS is infinite. Therefore y must be continuous at $t = 0$, and we can use the initial condition $y(0) = 0$.

Step Response Example

Step 1: set $f(t) = 1$, and solve for just $t \geq 0$.

$$\frac{dy}{dt} + 5y = 1$$

Complementary function: $\dot{y} + 5y = 0 \Rightarrow y = Ae^{-5t}$

Particular Integral: try $y = \lambda$ (some constant) $\Rightarrow y = \frac{1}{5}$

General Solution: $y = Ae^{-5t} + \frac{1}{5}$

Step 2: set the boundary condition $y = 0$ at $t = 0$

$y(0) = 0 \Rightarrow A + \frac{1}{5} = 0 \Rightarrow A = -\frac{1}{5}$

So step response is $y(t) = \frac{1}{5}\left(1 - e^{-5t}\right)$ for $t \geq 0$.

Step $\longrightarrow$ Impulse Response

$$\text{\textbf{Impulse Response} } g(t) \;\xrightarrow{\text{integrate}}\; \text{\textbf{Step Response}}$$

$$\text{\textbf{Impulse Response} } g(t) \;\xleftarrow{\text{differentiate}}\; \text{\textbf{Step Response}}$$

Step response is $y(t) = \frac{1}{5}\left(1 - e^{-5t}\right)$ for $t \geq 0$.

Impulse response $g(t)$ is given by:

$$g(t) = \begin{cases} 0, \ t < 0 \\ \dfrac{d}{dt}\left[\dfrac{1}{5}\left(1 - e^{-5t}\right)\right] = e^{-5t} & t \geq 0 \end{cases}$$

Find the Impulse Response

$$\frac{d^2y}{dt^2} + 13\frac{dy}{dt} + 12y = f(t)$$

1. Find the General Solution with $f(t) = 1$

Complementary function is $y = Ae^{-12t} + Be^{-t}$

Particular integral is $y = \frac{1}{12}$

General solution is $y = \frac{1}{12} + Ae^{-12t} + Be^{-t}$

2. Set boundary conditions $y(0) = \dot{y}(0) = 0$ to get the step response.

$$\begin{aligned}
&\tfrac{1}{12} + A + B = 0 \\
&-12A - B = 0 \\
&\Rightarrow A = \tfrac{1}{132} \text{ and } B = -\tfrac{1}{11}
\end{aligned}$$

Thus Step Response is $y = \frac{1}{12} + \frac{e^{-12t}}{132} - \frac{e^{-t}}{11}$

3. Differentiate the step response to get the impulse response.

$$g(t) = \frac{dy}{dt} = \frac{e^{-t} - e^{-12t}}{11} \qquad (t \geq 0)$$

Using the Impulse Response

If we have a system input composed of impulses,

$$f(t) = 3\delta(t-1) + 4\delta(t-2)$$

we can find the corresponding system output using superposition.

$$y(t) = 3g(t-1) + 4g(t-2)$$

Thus:

$$= \begin{cases} 0 & t < 1 \\ 3\left[\frac{e^{-(t-1)} - e^{-12(t-1)}}{11}\right] & 1 \le t \le 2 \\ 3\left[\frac{e^{-(t-1)} - e^{-12(t-1)}}{11}\right] + 4\left[\frac{e^{-(t-2)} - e^{-12(t-2)}}{11}\right] & t > 2 \end{cases}$$

More General Input

Suppose our input is composed of lots of delta functions:

$$f(t) = \sum_n p_n \delta(t - q_n)$$

Then the corresponding system output will be

$$y(t) = \sum_n p_n g(t - q_n)$$

Section 2: Summary

Differential Equation

$a\ddot{y} + b\dot{y} + cy + d = f(t)$

$\downarrow$

solve
$a\ddot{y} + b\dot{y} + cy + d = 1$
with boundary conditions
$y(0) = 0$ and $\dot{y}(0) = 0$

$\downarrow$

Step Response

$\downarrow$

differentiate

$\downarrow$

Impulse Response

$\downarrow$

Any input $\longrightarrow$ **convolution** $\longrightarrow$ **Corresponding output**

Section 3

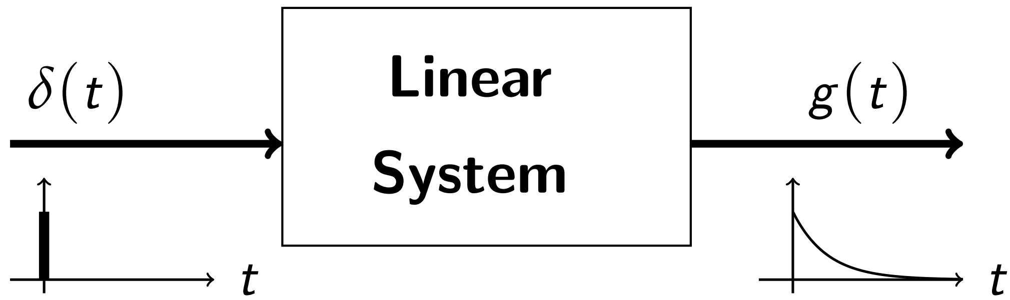
Convolution

In this section we derive the convolution integral and show its use in some examples.

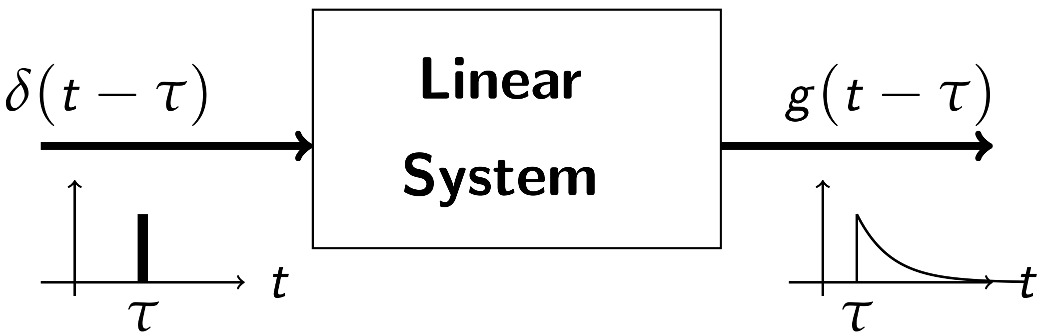
Convolution

Our goal is to calculate the output, $y(t)$, of a linear system (and we will assume it is an LTI system) using the input, $f(t)$, and the impulse response of the system, $g(t)$.

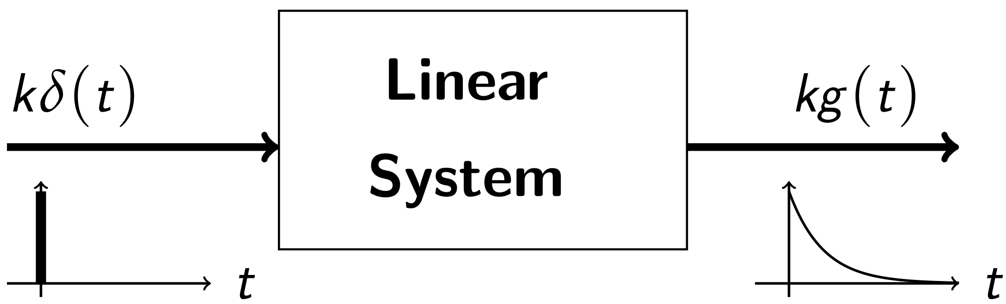
An impulse at time $t = 0$ produces the impulse response.



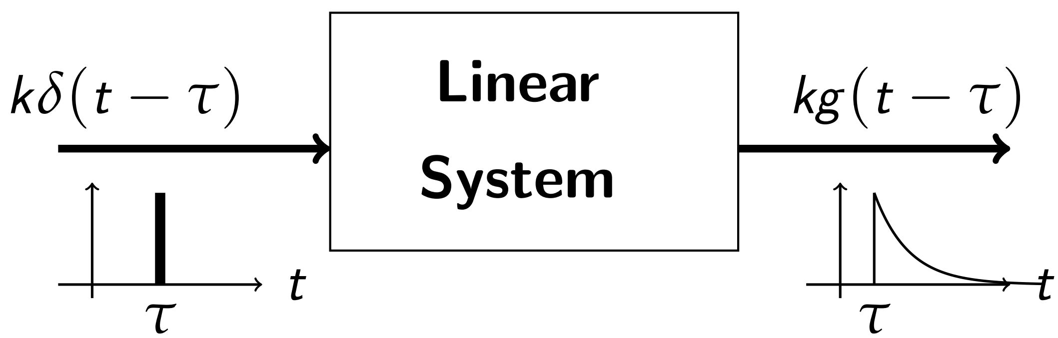
An impulse delayed to time $t = \tau$ produces a delayed impulse response starting at time τ.



A scaled impulse at time $t = 0$ produces a scaled impulse response.



An impulse that has been scaled by k and delayed to time $t = \tau$ produces an impulse response scaled by k and starting at time τ.



The following sketches the derivation of the convolution integral.
From the sifting property of the delta function (see earlier), we know that we can write an input $f(t)$ as

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\ d\tau \ = \ \lim_{\Delta\tau\to 0} \sum_n f(n\Delta\tau)\delta(t-n\Delta\tau)\Delta\tau$$

ie we write it as a *sum* of scaled and shifted delta functions. Therefore our output is a *sum* of scaled and shifted impulse responses.

$$y(t) = \lim_{\Delta\tau\to 0} \sum_n f(n\Delta\tau)g(t-n\Delta\tau)\Delta\tau = \int_{-\infty}^{\infty} g(t-\tau)f(\tau)d\tau$$

Our general convolution integral, giving the output y given the input f, is therefore:

$$y(t) = \int_{-\infty}^{\infty} g(t-\tau)f(\tau)d\tau \tag{1}$$

If we require that there is no response for $t < 0$, which implies $g(t-\tau) = 0$ for $t - \tau < 0$, ie for $\tau > t$, then we can rewrite equation 1 as:

$$y(t) = \int_{-\infty}^{t} g(t-\tau) f(\tau) d\tau \tag{2}$$

In addition, if we have a causal input, ie $f(t) = 0$ for $t < 0$, this reduces to

$$y(t) = \int_{0}^{t} g(t-\tau) f(\tau) d\tau \tag{3}$$

Equations 1,2,3 are all examples of **the convolution integral**. Note that equation 1 is the more general integral form, which reduces to equation 2 if we have no response for $t < 0$ and to equation 3 if both f and g are zero for $t < 0$.

To summarise:

$$y(t) = \int_{-\infty}^{\infty} g(t-\tau) f(\tau) d\tau$$

or

$$y(t) = \int_{-\infty}^{t} g(t-\tau) f(\tau) d\tau$$

or

$$y(t) = \int_{0}^{t} g(t-\tau) f(\tau) d\tau$$

Points to note:

- Treat t as a constant when evaluating the integral. The integration variable is τ.
- t is time as it relates to the output of the system $y(t)$.
- τ is time as it relates to the input of the system $f(\tau)$.

Convolution Example 1

Consider a system with impulse response

$$g(t) = \begin{cases} 0 & t < 0 \\ e^{-5t} & t \geq 0 \end{cases}$$

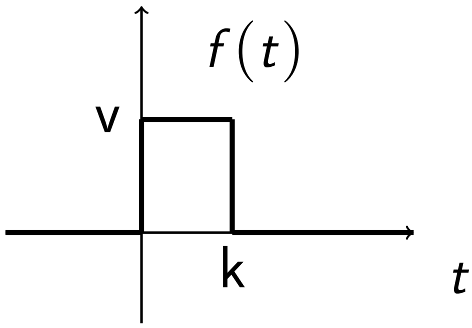
Find the output for input $f(t) = H(t)$ (step function).

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} g(t-\tau) f(\tau) d\tau \\ &= \int_{-\infty}^{t} e^{-5(t-\tau)} H(\tau) d\tau \\ &= \int_{0}^{t} e^{-5(t-\tau)} d\tau \\ &= \left[\frac{1}{5} e^{-5(t-\tau)} \right]_0^t \\ &= \frac{1}{5}\left(1 - e^{-5t}\right) \end{aligned}$$

Convolution Example 2

For the same system ($g(t) = e^{-5t}, \;\; t \geq 0$), find the output for input

$$f(t) = \begin{cases} 0 & t < 0 \\ v & 0 < t < k \\ 0 & t > k \end{cases}$$



Using the convolution integral, the answer is given by

$$y(t) = \int_{-\infty}^{\infty} g(t-\tau) f(\tau) d\tau = \int_{-\infty}^{t} g(t-\tau) f(\tau) d\tau$$

$$= \begin{cases} \int_{-\infty}^{t} g(t-\tau) \times 0 \, d\tau & t < 0 \\[2em] \int_{-\infty}^{0} g(t-\tau) \times 0 \, d\tau \\ \quad + \int_{0}^{t} g(t-\tau) \, v \, d\tau & 0 < t < k \\[2em] \int_{-\infty}^{0} g(t-\tau) \times 0 \, d\tau \\ \quad + \int_{0}^{k} g(t-\tau) \, v \, d\tau \\ \quad + \int_{k}^{t} g(t-\tau) \times 0 \, d\tau & t > k \end{cases}$$

Case (a): $t < 0$

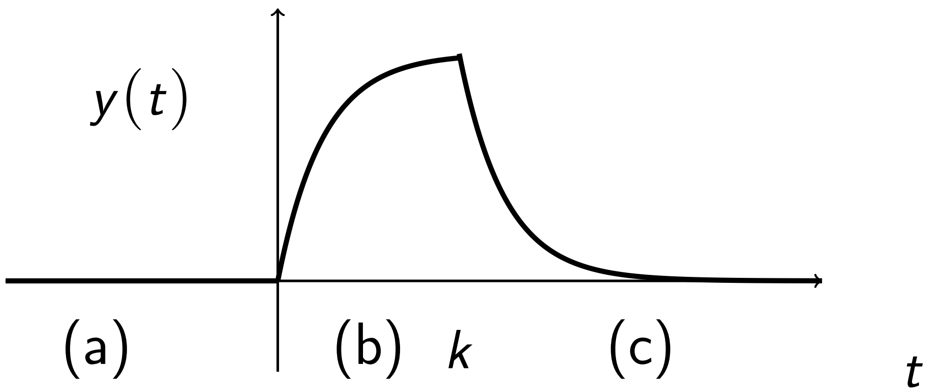
$\int_{-\infty}^{t} g(t-\tau) \times 0 \, d\tau = 0$ so $y(t) = 0$ for all $t < 0$.

Case (b): $0 < t < k$

$$\begin{aligned} y(t) = \int_0^t g(t-\tau)\, v \, d\tau &= \int_0^t e^{-5(t-\tau)}\, v \, d\tau \\ &= \frac{v}{5}\left[e^{-5(t-\tau)}\right]_0^t \\ &= \frac{v}{5}\left(1 - e^{-5t}\right) \end{aligned}$$

Case (c): $t > k$

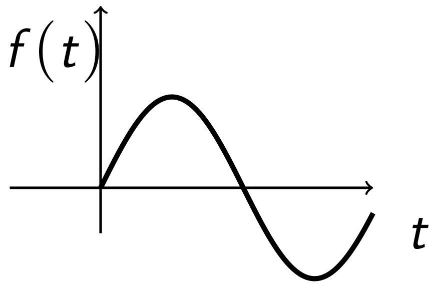
$$\begin{aligned} y(t) = \int_0^k g(t-\tau)\, v \, d\tau &= \int_0^k e^{-5(t-\tau)}\, v \, d\tau \\ &= \frac{v}{5}\left[e^{-5(t-\tau)}\right]_0^k \\ &= \frac{v}{5}\left(e^{5k} - 1\right) e^{-5t} \end{aligned}$$



Convolution Example 3

For the same system ($g(t) = e^{-5t}$, $t \geq 0$), find the output for input

$$f(t) = \begin{cases} 0 & t < 0 \\ \sin(\omega t) & t \geq 0 \end{cases}$$



Using the convolution integral, the answer is given by

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} g(t-\tau) f(\tau) d\tau = \int_{-\infty}^{t} g(t-\tau) f(\tau) d\tau \\ &= \begin{cases} \int_{-\infty}^{t} g(t-\tau) \times 0 \, d\tau & t < 0 \\ \\ \int_{-\infty}^{0} g(t-\tau) \times 0 \, d\tau \\ \quad + \int_{0}^{t} g(t-\tau) \sin(\omega\tau) \, d\tau & t > 0 \end{cases} \end{aligned}$$

Case (a): $t < 0$

$\int_{-\infty}^{t} g(t-\tau) \times 0 \, d\tau = 0$ so $y(t) = 0$ for all $t < 0$.

Case (b): $t > 0$

$$
\begin{aligned}
y(t) &= \int_0^t g(t-\tau)\sin(\omega\tau)\,d\tau \\
&= \int_0^t e^{-5(t-\tau)}\sin(\omega\tau)\,d\tau \\
&= \mathrm{Im}\left\{\int_0^t e^{-5(t-\tau)}e^{i\omega\tau}\,d\tau\right\} \\
&= \mathrm{Im}\left\{e^{-5t}\left[\frac{e^{(5+i\omega)\tau}}{5+i\omega}\right]_0^t\right\} \\
&= \mathrm{Im}\left\{\frac{e^{i\omega t}-e^{-5t}}{5+i\omega}\right\} \\
&= \frac{5\sin(\omega t)-\omega\cos(\omega t)+\omega e^{-5t}}{25+\omega^2}
\end{aligned}
$$

[**Note**: the output contains only terms of the same frequency as the input]

Convolution: Summary

Differential Equation

$$a\ddot{y} + b\dot{y} + cy + d = f(t)$$

$$\downarrow$$

solve

$$a\ddot{y} + b\dot{y} + cy + d = 1$$

with boundary conditions

$$y(0) = 0 \quad \text{and} \quad \dot{y}(0) = 0$$

$$\downarrow$$

Step Response

$$\downarrow$$

differentiate

$$\downarrow$$

Impulse Response: $g(t)$

$$\downarrow$$

Any input: $f(t)$ $\longrightarrow$ **convolution** $\longrightarrow$ **Corresponding output:** $y(t)$

$$y(t) = \int_{-\infty}^{\infty} g(t-\tau)\, f(\tau)\, d\tau$$

Complete Example

Find the impulse response of

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = f(t)$$

hence find the output when the input $f(t) = H(t)e^{-t}$.

1. Find the General Solution with $f(t) = 1$

Complementary function is $y = Ae^{-t} + Be^{-2t}$

Particular integral is $y = \frac{1}{2}$

General solution is $y = \frac{1}{2} + Ae^{-t} + Be^{-2t}$

2. Set boundary conditions $y(0) = \dot{y}(0) = 0$ to get the step response.

$$\begin{aligned} &\tfrac{1}{2} + A + B = 0 \\ &-A - 2B = 0 \\ &\Rightarrow A = -1 \text{ and } B = \tfrac{1}{2} \end{aligned}$$

Thus Step Response is $y = \frac{1}{2} - e^{-t} + \frac{e^{-2t}}{2}$

3. Differentiate the step response to get the impulse response.

$$g(t) = \frac{dy}{dt} = e^{-t} - e^{-2t}$$

4. Use the convolution integral to find the output for the required input.

The required input is $f(t) = e^{-t}$, $t > 0$.

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} g(t-\tau) f(\tau) d\tau = \int_{-\infty}^{t} g(t-\tau) f(\tau) d\tau \\
&= \int_0^t \left(e^{-(t-\tau)} - e^{-2(t-\tau)} \right) e^{-\tau} \, d\tau \\
&= \int_0^t e^{-t} - e^{\tau - 2t} \, d\tau \\
&= \left[\tau e^{-t} - e^{\tau - 2t} \right]_0^t \\
&= (t-1) \, e^{-t} + e^{-2t}
\end{aligned}$$

Section 3: Summary

Convolution integral (memorise this):

$$
\begin{aligned}
f(t) &= \text{input} \\
g(t) &= \text{impulse response} \\
y(t) &= \text{output} \\
y(t) &= \int_{-\infty}^{\infty} g(t-\tau)\, f(\tau)\, d\tau \\
&\textit{or} \\
y(t) &= \int_{-\infty}^{t} g(t-\tau) f(\tau) d\tau \\
&\textit{or} \\
y(t) &= \int_{0}^{t} g(t-\tau) f(\tau) d\tau
\end{aligned}
$$

Way to find the output of a linear system, described by a differential equation, for an arbitrary input:

- Find general solution to equation for input $= 1$.
- Set boundary conditions $y(0) = \dot{y}(0) = 0$ to get the step response.
- Differentiate to get the impulse response.
- Use convolution integral together with the impulse response to find the output for any desired input.

Section 4

Evaluating Convolution Integrals

A way of rearranging the convolution integral is described and illustrated.

The differences between convolution in time and space are discussed and the concept of causality is introduced (although we have already seen this).

The concept of a spatially-varying impulse is introduced and the section ends with an example of spatial convolution with a spatially-varying impulse response.

Convolution: Summary

Differential Equation

$$a\ddot{y} + b\dot{y} + cy + d = f(t)$$

$$\downarrow$$

$$\boxed{\begin{array}{c}\textbf{solve}\\ a\ddot{y} + b\dot{y} + cy + d = 1\\ \textbf{with boundary conditions}\\ y(0) = 0 \ \text{ and } \ \dot{y}(0) = 0\end{array}}$$

$$\downarrow$$

Step Response

$$\downarrow$$

$$\boxed{\textbf{differentiate}}$$

$$\downarrow$$

Impulse Response: $g(t)$

$$\downarrow$$

Any input: $f(t)$ $\longrightarrow$ $\boxed{\textbf{convolution}}$ $\longrightarrow$ **Corresponding output:** $y(t)$

$$y(t) = \int_{-\infty}^{\infty} g(t-\tau)\, f(\tau)\, d\tau$$

Splitting up Integrals

Suppose we have a function:

$$f(t) = \begin{cases} a & t < 0 \\ b & 0 < t < k \\ c & t > k \end{cases}$$

and we want to evaluate the integral $\int_{-\infty}^{t} f(\tau)\, d\tau$, we can split it up as follows:

$$\begin{array}{ll} \int_{-\infty}^{t} a\, d\tau & t < 0 \\ \int_{-\infty}^{0} a\, d\tau + \int_{0}^{t} b\, d\tau & 0 < t < k \\ \int_{-\infty}^{0} a\, d\tau + \int_{0}^{k} b\, d\tau + \int_{k}^{t} c\, d\tau & t > k \end{array}$$

Example

Find the impulse response of

$$\frac{d^2y}{dt^2} + 9y = f(t)$$

hence find the output for (i) input $f(t) = t,\ t > 0$ and (ii) input $f(t) = H(t) - H(t-1)$ (ie a pulse).

1. Find the General Solution with $f(t) = 1\ (t \geq 0)$.

Complementary function is $y = A\cos(3t) + B\sin(3t)$

Particular integral is $y = \frac{1}{9}$

General solution is $y = \frac{1}{9} + A\cos(3t) + B\sin(3t)$

2. Set boundary conditions $y(0) = \dot{y}(0) = 0$ to get the step response.

$$\begin{aligned} &\tfrac{1}{9} + A = 0 \\ &3B = 0 \\ &\Rightarrow A = -\tfrac{1}{9} \text{ and } B = 0 \end{aligned}$$

Thus the Step Response is

$$y = \frac{1}{9}(1 - \cos(3t))$$

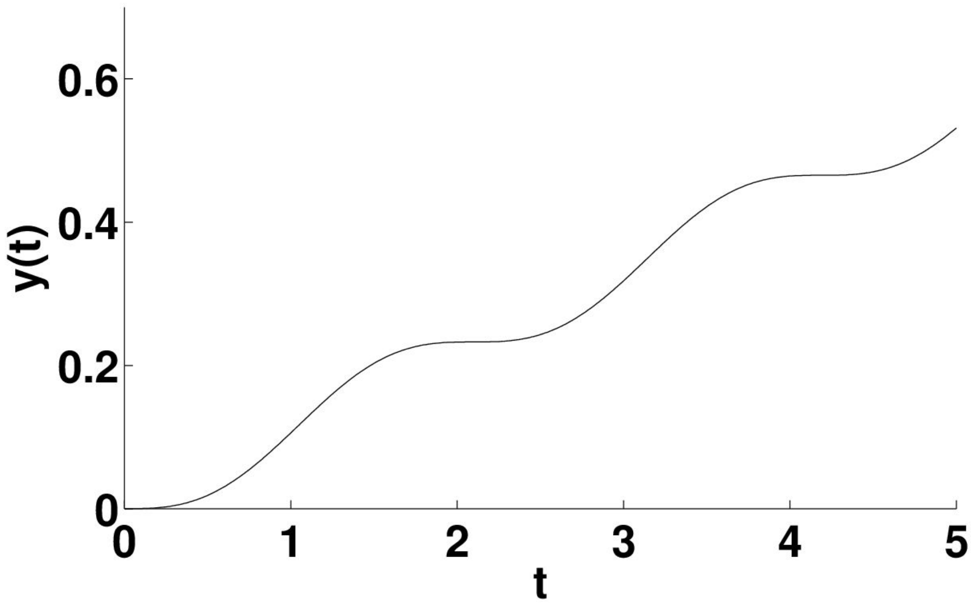
3. Differentiate the step response to get the impulse response.

$$g(t) = \frac{dy}{dt} = \frac{1}{3}\sin(3t)$$

4. Use the convolution integral to find the output for the required input.

For part (i) the required input is a ramp starting at the origin: $f(t) = t$ when $t > 0$ and $f(t) = 0$ otherwise.

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} g(t-\tau) f(\tau)\, d\tau = \int_0^t g(t-\tau) f(\tau)\, d\tau \\ &= \int_0^t \frac{1}{3}\sin(3(t-\tau)) \times \tau\, d\tau \\ &= \frac{t}{9} - \frac{\sin(3t)}{27} \end{aligned}$$



For part (ii) the required input is a pulse of unit height and unit duration: $f(t) = H(t) - H(t-1)$.

$$y(t) = \int_{-\infty}^{\infty} g(t-\tau) f(\tau) d\tau = \int_{-\infty}^{t} g(t-\tau) f(\tau) d\tau$$

$$= \begin{cases} \int_{-\infty}^{t} g(t-\tau) \times 0 \, d\tau & t < 0 \\ \\ \int_{-\infty}^{0} g(t-\tau) \times 0 \, d\tau & \\ \quad + \int_{0}^{t} g(t-\tau) \times 1 \, d\tau & 0 < t < 1 \\ \\ \int_{-\infty}^{0} g(t-\tau) \times 0 \, d\tau & \\ \quad + \int_{0}^{1} g(t-\tau) \times 1 \, d\tau & \\ \quad + \int_{1}^{t} g(t-\tau) \times 0 \, d\tau & t > 1 \end{cases}$$

Case (a): $t < 0$

$\int_{-\infty}^{t} g(t-\tau) \times 0 \, d\tau = 0$ so $y(t) = 0$ for all $t < 0$.

Case (b): $0 < t < 1$

$$
\begin{aligned}
y(t) &= \int_0^t g(t-\tau) \times 1 \, d\tau = \int_0^t \frac{1}{3} \sin(3(t-\tau)) \, d\tau \\
&= \frac{1}{9}(1 - \cos(3t))
\end{aligned}
$$

Case (c): $t > 1$

$$
\begin{aligned}
y(t) &= \int_0^1 g(t-\tau) \times 1 \, d\tau = \int_0^1 \frac{1}{3} \sin(3(t-\tau)) \, d\tau \\
&= \left[\frac{1}{9} \cos(3(t-\tau)) \right]_0^1 \\
&= \frac{1}{9} \{ \cos(3(t-1)) - \cos(3t) \}
\end{aligned}
$$

Part (ii) Another Way

The input for part (ii) is composed of two step functions. We can therefore calculate the output using the step response, $r(t) = \frac{1}{9}(1 - \cos(3t))$.

$$\text{Input} = H(t) - H(t-1) \Rightarrow \text{Output} = r(t) - r(t-1)$$

Hence, for $t > 1$,

$$\begin{aligned} y(t) &= \frac{1}{9}(1 - \cos(3t)) - \frac{1}{9}(1 - \cos(3(t-1))) \\ &= \frac{1}{9}\{\cos(3(t-1)) - \cos(3t)\} \end{aligned}$$

Alternative Convolution Integral

The normal convolution integral

$$y(t) = \int_{-\infty}^{\infty} g(t-\tau)\, f(\tau)\, d\tau$$

can be inconvenient to compute when we have a complicated expression for $g(t)$.

We would therefore like to derive an alternative version of the convolution integral that has a term of the form $g(\tau)$ rather than $g(t-\tau)$ as this will be easier to calculate in cases where g is a complicated expression.

Arguments of f and g

Substitute $u = t - \tau$ in the convolution formula. We have $-du = d\tau$,

$$\int_{-\infty}^{\infty} g(t-\tau)f(\tau)d\tau = -\int_{\infty}^{-\infty} g(u)f(t-u)du$$
$$= \int_{-\infty}^{\infty} g(u)f(t-u)du$$

As u is the variable of integration, we can call it anything, as it disappears when the integration has been evaluated. We therefore choose to rename u as τ. Hence:

$$\int_{\infty}^{\infty} g(t-\tau)f(\tau)d\tau = \int_{-\infty}^{\infty} g(\tau)f(t-\tau)d\tau$$

Note: if both functions are zero for $t < 0$, we can also write

$$\int_{0}^{t} g(t-\tau)f(\tau)d\tau = \int_{0}^{t} g(\tau)f(t-\tau)d\tau$$

So it does not matter which way round we have the arguments to the functions in the convolution integral. (Causal form requires both functions to be zero for $t < 0$).

Example

Consider a linear system with impulse response

$$g(t) = \begin{cases} 3t^2 - 4t + 7 & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the output for the input $f(t) = t$, $(t \geq 0)$ and $f(t) = 0$, $(t < 0)$.

Note that everything is zero for $t < 0$, so that

$y(t) = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)\,d\tau \; = \; \int_0^t f(t-\tau)g(\tau)\,d\tau.$

$$\begin{aligned} y(t) &= \int_0^t f(t-\tau)g(\tau)\,d\tau \\ &= \int_0^t (t-\tau) \times (3\tau^2 - 4\tau + 7)\,d\tau \\ &= \frac{t^4}{4} - \frac{2t^3}{3} + \frac{7t^2}{2} \end{aligned}$$

Spatial Convolution

Systems with time-varying input & output.

Causal: no output before the input that causes it.

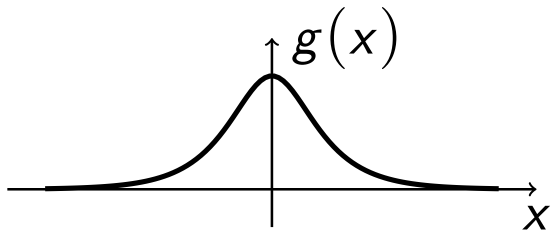
$$g(t) = 0, \; t < 0$$

Systems with input, output a function of position.

An input can affect the output on either side. $g(x)$ can be non-zero for any x.

Consider a one-dimensional strip of a material that is known to deform linearly according to

$$g(x) = \frac{1}{\cosh(x)}$$

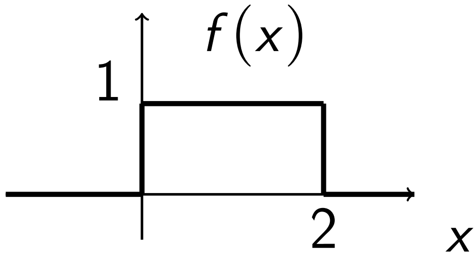


when subject to a unit force at $x = 0$.

This is a spatial impulse response.

Spatial Convolution Example

Calculate the deformation of a strip of material with spatial impulse response as described on the previous page in response to a uniform load of $f(x) = 1$ applied from $x = 0$ to $x = 2$.



Use a rather than τ as the integration variable:

$$
\begin{aligned}
y(x) &= \int_{-\infty}^{\infty} g(x-a) f(a)\, da \\
&= \int_{-\infty}^{0} \frac{1}{\cosh(x-a)} \times 0\, da \\
&\quad + \int_{0}^{2} \frac{1}{\cosh(x-a)} \times 1\, da \\
&\quad + \int_{2}^{\infty} \frac{1}{\cosh(x-a)} \times 0\, da \\
&= \int_{0}^{2} \frac{1}{\cosh(x-a)}\, da \\
&= 2\left\{\arctan\left(e^{2-x}\right) - \arctan\left(e^{-x}\right)\right\}
\end{aligned}
$$

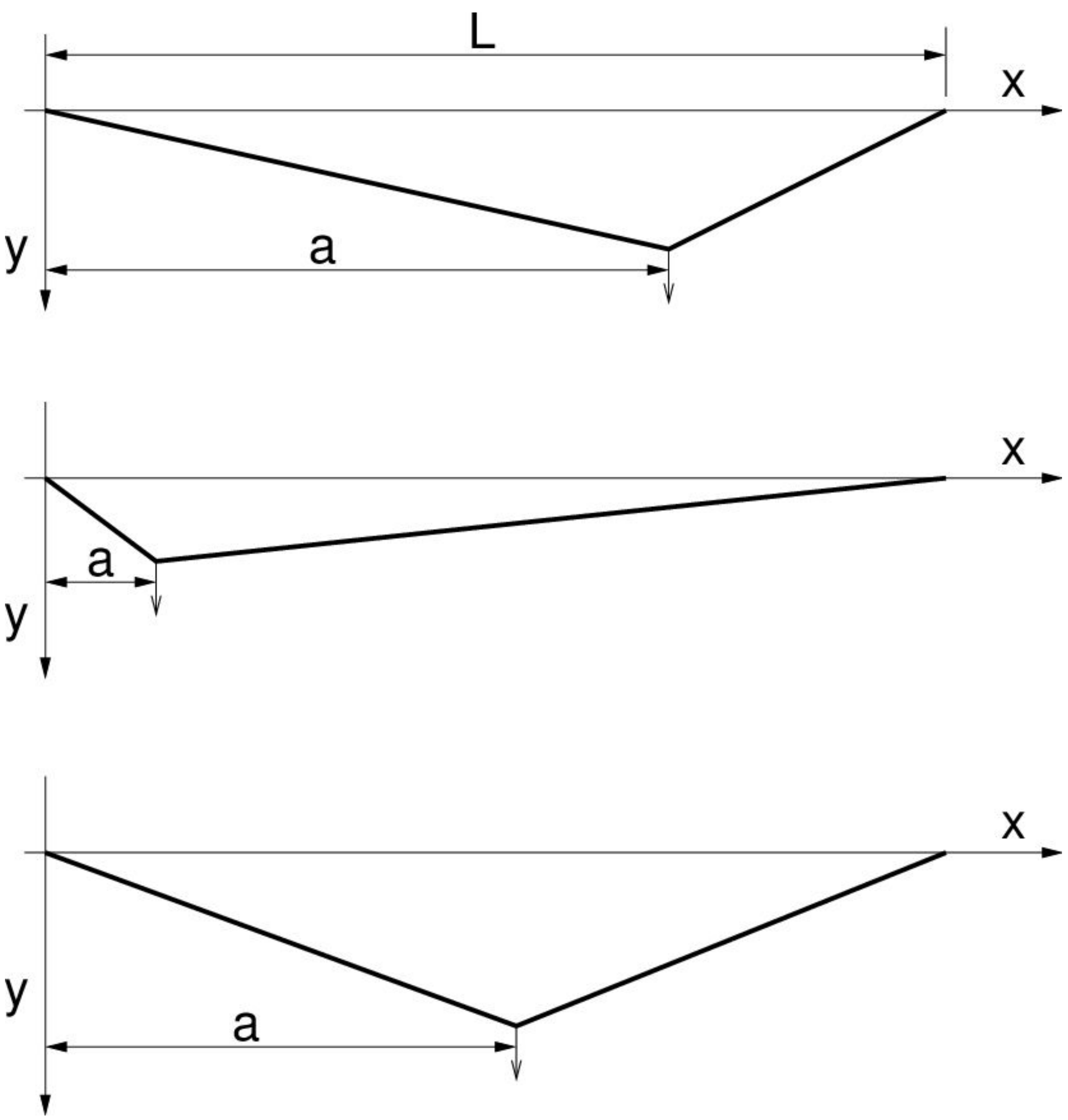
Variable Impulse Response

Consider a taut string suspended between two points a distance L apart. It is subject to a uniform loading of K per unit length which results in a *small* displacement.

If we knew the deformation caused by a point load, we could integrate in a style similar to the convolution integral to find the shape under the distributed load.

If it was possible to have a spatial impulse response $g(x)$ then we could say $y(x) = \int_0^L g(x-a)F(a)\,da$

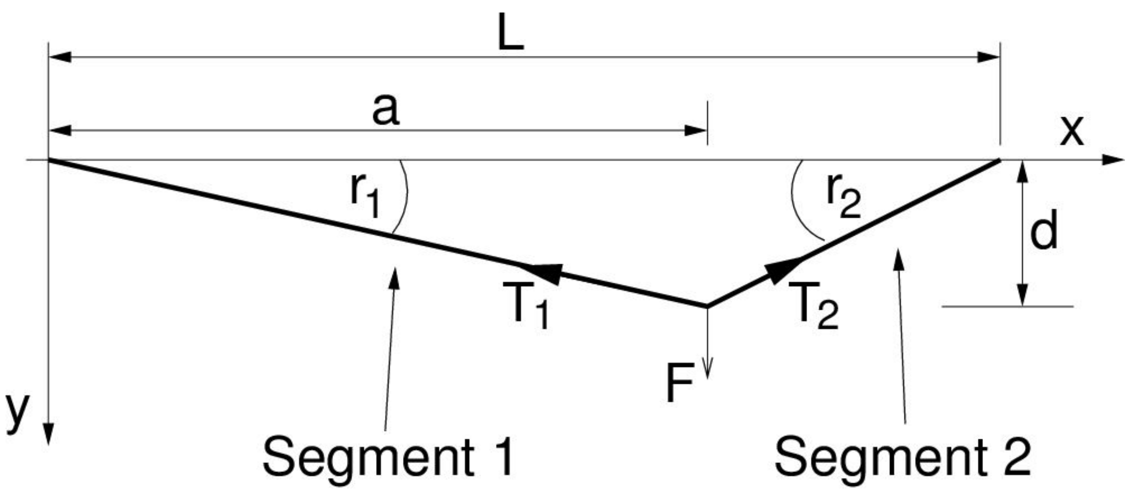
where F is the loading.



But a normal impulse response is not possible because the shape of g changes depending on the position of the point load along the string. We have a function $g(x, a)$ where the point load is at position a. The function g that gives the displacement under a point load depends on both the position of the load, a, and the position at which you want to know the displacement, x.

If we can find this $g(x,a)$ we can work out the complete displacement under the continuous load K using

$$y(x) = \int_0^L g(x,a)F(a)\,da = \int_0^L g(x,a)K\,da$$



To find $g(x,a)$ we first work out the maximum displacement, d, for a point load, $F = 1$, at position a.

Resolve horizontally: $T_1 \cos(r_1) = T_2 \cos(r_2)$

Use the approximation: $\cos(r_1) \approx \cos(r_2) \approx 1$
[for small displacements]

This gives us: $T_1 = T_2$. Call this tension T.

Now resolve vertically: $T(\sin(r_1) + \sin(r_2)) = 1$

Again approximate: $\cos(r_1) \approx \cos(r_2) \approx 1$
so that $\tan(r_i) \approx \cos(r_i)$

This gives us: $T(\tan(r_1) + \tan(r_2)) = 1$

$$\Rightarrow \left(\frac{d}{a} + \frac{d}{L-a}\right) = \frac{1}{T}$$

$$\Rightarrow \frac{Ld}{a(L-a)} = \frac{1}{T}$$

$$\text{so} \qquad d = \frac{a(L-a)}{TL}$$

This enables us to write down equations for the two straight segments of the $g(x, a)$ function.

Segment 1: $x < a$

$$g(x,a) = \left(\frac{x}{a}\right) d = \frac{x}{a}\left[\frac{a(L-a)}{TL}\right] = \frac{x(L-a)}{TL}$$

Segment 2: $x > a$

$$g(x,a) = \left(\frac{L-x}{L-a}\right) d = \frac{L-x}{L-a}\left[\frac{a(L-a)}{TL}\right] = \frac{a(L-x)}{TL}$$

Finally, we work out the shape of a string of length L with tension T under a uniform load of K per unit length.

$$\begin{aligned}
y(x) &= \int_0^L g(x,a)F(a)\,da \\
&= \int_0^L g(x,a)K\,da \\
&= \int_0^x g^{\boxed{\text{seg 2}}} \times K\,da + \int_x^L g^{\boxed{\text{seg 1}}} \times K\,da \\
&= \int_0^x \frac{a(L-x)K}{TL}\,da + \int_x^L \frac{x(L-a)K}{TL}\,da \\
&= \left(\frac{K}{2T}\right) x(L-x)
\end{aligned}$$

Note: the above holds because when viewed as a function of a, $g(x,a)$ is given by

$$g(x,a) = \begin{cases} \frac{a(L-x)}{TL} & a < x \\ \frac{x(L-a)}{TL} & a > x \end{cases}$$

Section 4: Summary

The convolution integral is:

$$\int_{-\infty}^{\infty} g(t-\tau)f(\tau)d\tau = \int_{-\infty}^{\infty} g(\tau)f(t-\tau)d\tau$$

If $f(t) = g(t) = 0$ for all $t < 0$ then

$$\int_{0}^{t} g(t-\tau)f(\tau)d\tau = \int_{0}^{t} g(\tau)f(t-\tau)d\tau$$

Systems for which $g(t) = 0$ for $t < 0$ are called *causal* systems.

Systems with time-varying inputs and outputs are causal.

Systems that have inputs and outputs that vary as a function of spatial location can have $g(x) \neq 0$ for any x.

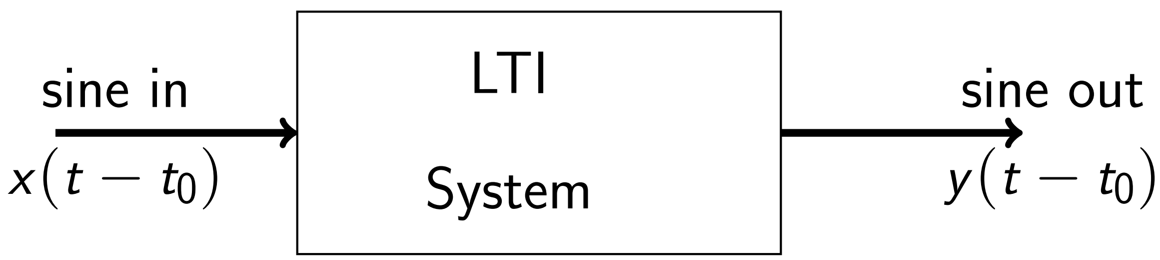
We have learnt how to handle a spatially-varying impulse response.

Selected bits of Sections 1-4

We discuss a few things here that require a little more explanation than was given on the first run-through (won't necessarily go through in lectures).

LTI Systems

Recall that in Section 1 we asserted that if we define an LTI system via the property that a linear system which behaves such that a shifted input produces a shifted output, then an equivalent definition is that *sine in* $\Longrightarrow$ *sine out*, ie an input of a sine wave of frequency ω produces an output which may have a phase and amplitude change, but no frequency change.



Suppose, for example, that an input of $\sin(\omega t)$ is passed through an LTI system and produces an output of $\sin(\omega t) + \sin(2\omega t)$, ie that the frequency of the input is not preserved in the output:

$$\sin(\omega t) \quad \longrightarrow \quad \sin(\omega t) + \sin(2\omega t)$$

Now we know that the periodic nature of sine means that (where n is odd)

$$\sin(\omega t + n\pi) \equiv \sin(\omega(t + n\pi/\omega)) = -\sin(\omega t)$$

But the shift invariance property of the LTI system implies that

$$\begin{aligned}
\sin(\omega t + n\pi) = -\sin(\omega t) \quad &\rightarrow \quad \sin(\omega t + n\pi) + \sin(2\omega(t + n\pi/\omega)) \\
&\rightarrow \quad -\sin(\omega t) + \sin(2\omega t + 2n\pi)) \\
&\rightarrow \quad -\sin(\omega t) + \sin(2\omega t))
\end{aligned}$$

But

$$\sin(\omega t + n\pi) = -\sin(\omega t) \quad \rightarrow \quad -\sin(\omega t) - \sin(2\omega t)$$

And the two above expressions are incompatible, thus giving us a contradiction. We therefore conclude that there can be no frequency component other than ω in the output if $\sin(\omega t)$ is input. (This proof by contradiction can be made more rigorous).