A multivariate generalization of Prony's method

Ulrich von der Ohe
joint with
Stefan Kunis Thomas Peter Tim Römer

Institute of Mathematics
Osnabrück University

SPARS 2015
Signal Processing with Adaptive Sparse Structured Representations
Cambridge
July 8, 2015

Outline

d-variate exponential sums

$f_1, \ldots, f_M \in \mathbb{C} \setminus \{0\}$, pairwise distinct $z_1, \ldots, z_M \in (\mathbb{C} \setminus \{0\})^d$, then

$$\begin{aligned} f : \mathbb{Z}^d &\longrightarrow \mathbb{C} \\ k &\longmapsto \sum_{j=1}^{M} f_j z_j^k = \sum_{j=1}^{M} f_j \prod_{\ell=1}^{d} z_{j,\ell}^{k_\ell} \end{aligned}$$

is a *d-variate exponential sum*.

$$\Omega := \{z_1, \ldots, z_M\}.$$

Reconstruction problem

Task: Reconstruct all f_j and z_j from samples of f.

Solution for $d = 1$: de Prony (1795)

- $2M + 1$ samples $f(-M), \dots, f(M)$
- Reconstruct parameters z_j *independently* of coefficients f_j

Prony method for univariate exponential sums

$f : \mathbb{Z} \to \mathbb{C}$ univariate exponential sum, i. e. $d = 1$.

Classical Prony method:

Given: $f(-M), \ldots, f(M)$.

- $$T := (f(k-m))_{\substack{m=0,\ldots,M \\ k=0,\ldots,M}} \in \mathbb{C}^{M+1 \times M+1}.$$

  Fact: $\dim \ker T = 1$.

  $$(p_0, \ldots, p_M) \in \ker T \setminus \{0\}.$$

- z_j — roots of polynomial $p := \sum_{k=0}^{M} p_k \mathrm{X}^k \in \mathbb{C}[\mathrm{X}]$.
- $(f_1, \ldots, f_M) \in \mathbb{C}^M$ — solution to a regular linear system.

Literature (1795–2015)

For example

G. de Prony.
Essai expérimental et analytique: Sur les lois de la Dilatabilité de fluides élastiques et sur celles de la Force expansive de la vapeur de l'eau et de la vapeur de l'alkool, à différentes températures.
J. de l'École Polytechnique, 1:24–76, 1795.

G. Plonka and M. Tasche.
Prony methods for recovery of structured functions.
GAMM-Mitteilungen, 37(2):239–258, 2014.

Multivariate:

F. Andersson, M. Carlsson, and M. V. de Hoop.
Nonlinear approximation of functions in two dimensions by sums of exponential functions.
Appl. Comput. Harmon. Anal., 29:156–181, 2010.

E. J. Candès and C. Fernandez-Granda.
Towards a mathematical theory of super-resolution.
Comm. Pure Appl. Math., 67(6):906–956, 2013.

D. Potts and M. Tasche.
Parameter estimation for multivariate exponential sums.
Electron. Trans. Numer. Anal., 40:204–224, 2013.

T. Peter, G. Plonka, and R. Schaback.
Reconstruction of multivariate signals via Prony's method.
Submitted.

Here: Deterministic method, sampling set fixed a priori!

$d = 1$ revisited

Construct $T \in \mathbb{C}^{M+1 \times M+1}$ with

$$\ker T = \operatorname{span}\{p\}$$

and

$$\Omega = \{z \in \mathbb{C} \mid p(z) = 0\}.$$

Thus

$$\Omega = \{z \in \mathbb{C} \mid q(z) = 0 \text{ for all } q \in \ker T\} = \mathrm{V}(\ker T).$$

Multivariate Prony method

Now: $d \in \mathbb{N}$ arbitrary, so $f : \mathbb{Z}^d \to \mathbb{C}$.

Notation:

$$\Pi_n := \{q \in \mathbb{C}[X_1, \ldots, X_d] \mid \underbrace{\max\{\|k\|_\infty \mid k \in \mathbb{N}^d,\ q_k \neq 0\}}_{=:\mathrm{maxdeg}\, q} \leq n\},$$

$$N := \dim \Pi_n = (n+1)^d.$$

Idea: Identify Π_n with $\mathbb{C}^N$ and construct a matrix

$$T_n \in \mathbb{C}^{N \times N}$$

with

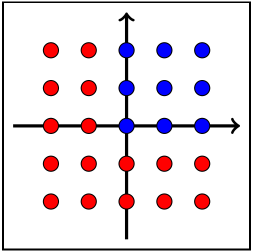
$$\mathrm{V}(\ker T_n) = \Omega.$$

Multivariate Prony method

For $n \in \mathbb{N}$ let

$$T_n := (f(k-m))_{\|m\|_\infty, \|k\|_\infty \le n} \in \mathbb{C}^{N \times N}$$

(block Toeplitz with Toeplitz blocks).



$d = n = 2$ sampling grid; $(2n+1)^2 = 25$ samples

Multivariate Prony method

Let

$$\mathcal{A}_n \colon \Pi_n \longrightarrow \mathbb{C}^M$$
$$p \longmapsto (p(z_1), \ldots, p(z_M))$$

be the evaluation homomorphism at $\Omega = \{z_1, \ldots, z_M\}$,

$$A_n \in \mathbb{C}^{M \times N}$$

the representation matrix of $\mathcal{A}_n$.

Then

$$T_n = P_n A_n^\top D_n A_n,$$

with $D_n := \operatorname{diag}(z_1^{-n} f_1, \ldots, z_M^{-n} f_M)$. Thus

$$\ker T_n \supseteq \ker A_n, \quad \text{so} \quad \mathrm{V}(\ker T_n) \subseteq \mathrm{V}(\ker A_n)$$

Remarks

So far

$$\mathrm{V}(\ker T_n) \subseteq \mathrm{V}(\ker \mathcal{A}_n) \supseteq \Omega.$$

Is $\mathrm{V}(\ker T_n) \neq \Omega$ possible?

Yes, $\mathrm{V}(\ker T_n)$ can be *infinite*, e. g. if $\dim \ker T_n = 1$.

So, how to choose n?

Multivariate polynomials $\rightsquigarrow$ algebraic tools!

$\mathrm{V}(\ker T_n) = \Omega$?

Theorem

Let f be a d-variate exponential sum with parameters $z_1, \ldots, z_M$. If $n \geq M$, then $\mathrm{V}(\ker T_n) = \Omega := \{z_1, \ldots, z_M\}$.

Proof.

Show:

- $\mathrm{V}(\ker \mathcal{A}_n) = \Omega$.
- $\operatorname{rank} A_n = M$, and thus $\ker A_n = \ker T_n$.

Example 1

$$\begin{aligned} f\colon \mathbb{Z}^2 &\longrightarrow \mathbb{C} \\ k &\longmapsto (1,1)^k + (-1,-1)^k. \end{aligned}$$

Choose $n = 2$. Then

$$T_2 = \begin{pmatrix} T & T' & T \\ T' & T & T' \\ T & T' & T \end{pmatrix}, \; T := \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}, \; T' := \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix},$$

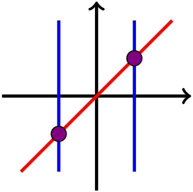
$$\dim \ker T_2 = 7 \text{ and } \langle \ker T_2 \rangle = \langle -1 + \mathsf{X}^2,\; -\mathsf{X} + \mathsf{Y} \rangle,$$

hence

$$\mathsf{V}(\ker T_2) = \mathsf{V}(-1 + \mathsf{X}^2,\; -\mathsf{X} + \mathsf{Y}) = \{(1,1),\; (-1,-1)\}.$$

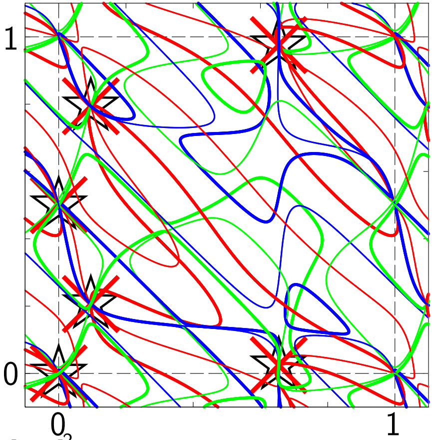
Example 1

Visually:



$$p_1 = -1 + X^2,$$
$$p_2 = -X + Y,$$
$$V(p_1, p_2) = \{(-1,-1),(1,1)\} = \Omega,$$

Example 2



Zero loci on $\mathbb{T}^2 \cong [0,1[^2$ of real- and imaginary parts of $\mathbb{C}$-basis of $\ker T_2$ for $M = 6$, $z_j = \exp(\pi i(1+\cos((j-1)\pi/2), 1+\sin((j-1)\pi/2)))$, $j = 1, \ldots, 5$, $z_5 = (1,1)$, $f_j = 1$, and $n = 2 < M$ (computed with `Octave` resp. `Bertini`)

Relation to dual certificate/TV-minimization

Theorem

Let $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$, *f M-sparse d-variate exponential sum with* $\Omega \subseteq \mathbb{T}^d$, $n \geq M$, *and* $\{p_1, \ldots, p_M\}$ *orthonormal basis of* $(\ker T_n)^\perp$. *Then*

$$\begin{aligned} q\colon [0,1[^d &\longrightarrow [0,1] \\ t &\longmapsto \frac{1}{N} \sum_{\ell=1}^{M} |p_\ell(\exp(2\pi \mathrm{i} t))|^2 \end{aligned}$$

is a trigonometric polynomial of degree n with

$$q(t) = 1 \quad \Longleftrightarrow \quad \exp(2\pi \mathrm{i} t) \in \Omega.$$

Summary

- Parameter reconstruction for d-variate exponential sums
  - Deterministic
  - Sampling set known a priori

- $T_n \in \mathbb{C}^{N \times N}$, $N = (n+1)^d$
- $\ker T_n \subseteq \mathbb{C}[\mathrm{X}_1, \ldots, \mathrm{X}_d]$
- $n \geq M \Rightarrow \mathrm{V}(\ker T_n) = \Omega$
- For $\Omega \subseteq \mathbb{T}^d$:
  Trig. poly. $q\colon [0,1[^d \to [0,1]$ with $q(t) = 1 \Leftrightarrow \exp(2\pi \mathrm{i} t) \in \Omega$

arxiv.org/abs/1506.00450

Summary

- Parameter reconstruction for d-variate exponential sums
  - Deterministic
  - Sampling set known a priori

- $T_n \in \mathbb{C}^{N \times N}$, $N = (n+1)^d$
- $\ker T_n \subseteq \mathbb{C}[\mathrm{X}_1, \dots, \mathrm{X}_d]$
- $n \geq M \Rightarrow \mathrm{V}(\ker T_n) = \Omega$
- For $\Omega \subseteq \mathbb{T}^d$:
  Trig. poly. $q\colon [0,1[^d \to [0,1]$ with $q(t) = 1 \Leftrightarrow \exp(2\pi \mathrm{i} t) \in \Omega$

arxiv.org/abs/1506.00450

Thank you for your attention!

References

D. J. Bates, J. D. Hauenstein, A. J. Sommese, and C. W. Wampler.
Bertini: Software for numerical algebraic geometry.
Available at bertini.nd.edu with permanent doi:
dx.doi.org/10.7274/R0H41PB5.

J. W. Eaton, D. Bateman, S. Hauberg, and R. Wehbring.
GNU Octave version 3.8.1 manual: a high-level interactive language for numerical computations.
CreateSpace Independent Publishing Platform, 2014.
ISBN 1441413006.