Exact Support Recovery for Sparse Spikes Deconvolution

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Joint work with Vincent Duval & Quentin Denoyelle

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Sparse Deconvolution

Neural spikes (1D)

\[ y = \varphi \ast m_0 + w \]

\( m_0 \) is “sparse”
Sparse Deconvolution

Neural spikes (1D)

Original Signal

Low-pass filter

Noise

Observation

\[ y = \varphi \ast m_0 + w \]

\( m_0 \) is “sparse”

Seismic imaging (1.5D)
Sparse Deconvolution

Neural spikes (1D)

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Original Signal

\[ y = \varphi \ast m_0 + w \]

Observation

\[ y = \varphi \ast m_0 + w \]

Seismic imaging (1.5D)

Astrophysics (2D)
Sparse Deconvolution

Neural spikes (1D)

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\( m_0 \) is “sparse”

Presented results extend to \( n \)D problems

Seismic imaging (1.5D)  Astrophysics (2D)
Overview

• Sparse Spikes Super-resolution

• Robust Support Recovery

• Asymptotic Positive Measure Recovery

• Discrete vs. Continuous
Radon measure $m$ on $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^d$. 

Discrete measure:

$$m_{a,x} = \sum_{i=1}^{N} a_i \delta_{x_i}, \ a \in \mathbb{R}^N, x \in \mathbb{T}^N$$
Radon measure $m$ on $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^d$.

Discrete measure:

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Linear measurements:

$$y = \Phi(m) + w \quad \varphi \in C^2(\mathbb{T} \times \mathbb{T})$$

$$\Phi(m) = \int_{\mathbb{T}} \varphi(x, \cdot)dm(x)$$
Radon measure $m$ on $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^d$.

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**Example:** 1-D ($d = 1$) convolution

$$\varphi(x, t) = \varphi(x - t)$$
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Example: 1-D ($d = 1$) convolution

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Minimum separation:

$$\Delta = \min_{i \neq j} |x_i - x_j|$$

$\rightarrow$ Signal-dependent recovery criteria.
Discrete $\ell^1$ regularization:

Computation grid $z = (z_k)_{k=1}^K$.

Basis-pursuit / Lasso:

$$\min_{a \in \mathbb{R}^N} \frac{1}{2} \|y - \Phi(m_{a,z})\|^2 + \lambda \|a\|_1$$
Sparse Deconvolution of Measures

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Grid-free regularization: total variation of measures:

$$|m|(\mathbb{T}) = \sup \left\{ \int \eta dm : \eta \in C(\mathbb{T}), \|\eta\|_\infty \leq 1 \right\}$$

For discrete measures: $|m_{a,z}|(\mathbb{T}) = \|a\|_1$. 
Discrete \( \ell^1 \) regularization:

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For discrete measures: \( |m_{a,z}|(\mathbb{T}) = \| a \|_1 \).

Sparse recovery:

\[
\min_m \frac{1}{2} \| \Phi(m) - y \|^2 + \lambda |m|(\mathbb{T}) \quad (\mathcal{P}_\lambda(y))
\]
Sparse Deconvolution of Measures

Discrete $\ell^1$ regularization:
Computation grid $z = (z_k)^K_{k=1}$.

Basis-pursuit / Lasso:

$$\min_{a \in \mathbb{R}^N} \frac{1}{2} ||y - \Phi(m_{a,z})||^2 + \lambda ||a||_1$$

Grid-free regularization: total variation of measures:

$$|m|(\mathbb{T}) = \sup \{ \int \eta dm : \eta \in C(\mathbb{T}), ||\eta||_{\infty} \leq 1 \}$$

For discrete measures: $|m_{a,z}|(\mathbb{T}) = ||a||_1$.

Sparse recovery: $\min_m \frac{1}{2} ||\Phi(m) - y||^2 + \lambda |m|(\mathbb{T})$ ($P_\lambda(y)$)

$\rightarrow$ Algorithms: [Bredies, Pikkarainen, 2010] (proximal-based)
[Candès, Fernandez-G. 2012] (root finding)
Dual Minimization

Primal program:

$$\min_m \frac{1}{2} \| \Phi(m) - y \|^2 + \lambda |m| (\mathcal{T})$$
Dual Minimization

Primal program:
\[
\min_m \frac{1}{2} \| \Phi(m) - y \|^2 + \lambda \| m \| (T)
\]

Dual program:
\[
\min \quad \| y/\lambda - p \|
\]
\[
\| \eta \|_\infty \leq 1
\]
\[
\eta = \Phi^* p
\]
**Dual Minimization**

**Primal program:**
\[
\min_m \frac{1}{2} \| \Phi(m) - y \|^2 + \lambda |m|(T)
\]

**Dual program:**
\[
\begin{align*}
\min \quad & \|y/\lambda - p\| \\
\text{subject to} \quad & \|\eta\|_\infty \leq 1 \\
\end{align*}
\]
\[
\eta = \Phi^* p
\]

**Proposition: [primal-dual relations]**
\[
\text{supp}(m) \subset \{ t ; |\eta(t)| = 1 \} \quad (1)
\]
\[
\eta = \lambda^{-1} \Phi^* (y - \Phi m) \quad (2)
\]
If $\Phi_x$ is injective, $\subset$ is $=$.

**Algorithm:** [Compute solution $m = m_{a,x}$]

*Step 1:* Compute $x = \text{supp}(m)$ using (1).

If $\Phi_x$ is injective, $\subset$ is $=$.
Primal program: 
\[
\min_m \frac{1}{2} \| \Phi(m) - y \|^2 + \lambda |m|(T)
\]

Dual program: 
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\min \| y/\lambda - p \|
\]
\[
\| \eta \|_\infty \leq 1 \quad \eta = \Phi^*p
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Algorithm: [Compute solution \( m = m_{a,x} \)]

Step 1: Compute \( x = \text{supp}(m) \) using (1).
If \( \Phi_x \) injective, \( \subset \) is =.

Step 2: Compute \( a \) using (2).
\[
a = \Phi_x^+ y - \lambda (\Phi_x^* \Phi_x)^{-1} \eta(x)
\]
Low frequency measurements:

\((\Phi^* p)(t) = \sum_{|k| \leq f_c} p_k e^{2i\pi kt}\) in 1D
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\[(\Phi^* p)(t) = \sum_{|k| \leq f_c} p_k e^{2i\pi kt}\] in 1D

\[
\min \left\{ \| y/\lambda - p \|^2 ; p \in C \right\} (\mathcal{D}_\lambda(y))
\]

\[C \overset{\text{def.}}{=} \{ p \in \mathbb{C}^P ; \| \Phi^* p \|_\infty \leq 1 \}\]
Low frequency measurements:

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\[\min \{ \|y/\lambda - p\|^2 ; p \in C \} \quad (D_\lambda(y))\]

\[C \overset{\text{def.}}{=} \{ p \in \mathbb{C}^P ; \|\Phi^* p\|_\infty \leq 1 \}\]

\[= \{ p ; 1 - \Phi^* p \text{ and } \Phi^* p - 1 \text{ are SOS} \}\]
SDP Dual Resolution and Root Finding

Low frequency measurements:
\[(\Phi^* p)(t) = \sum_{|k| \leq f_c} p_k e^{2i\pi kt}\text{ in 1D}\]

\[
\min \left\{ \|y/\lambda - p\|^2 ; \ p \in \mathcal{C} \right\} \quad \left( \mathcal{D}_\lambda(y) \right)
\]

\[
\mathcal{C} \overset{\text{def.}}{=} \left\{ p \in \mathbb{C}^P ; \ \|\Phi^* p\|_{\infty} \leq 1 \right\}
= \left\{ p ; 1 - \Phi^* p \text{ and } \Phi^* p - 1 \text{ are SOS} \right\}
= \left\{ p ; \exists Q \in \mathcal{H}, \begin{pmatrix} Q & p \\ p^* & 1 \end{pmatrix} \in S_N^+ \right\}
\]

\[
\mathcal{H} \overset{\text{def.}}{=} \left\{ Q \in \mathbb{C}^{P \times P} ; \ \sum_i Q_{i,i+j} = \delta_{i,j} \right\}
\]
Low frequency measurements:

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Algorithm: (i) Solve \( D_\lambda(y) \) (SDP program).
Low frequency measurements:

\[(\Phi^* p)(t) = \sum_{|k| \leq f_c} p_k e^{2i\pi kt}\]

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\[
\min \left\{ \|y/\lambda - p\|^2 ; p \in C \right\} \quad (D_\lambda(y))
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\]

\[\mathcal{H} \overset{\text{def.}}{=} \{ Q \in \mathbb{C}^{P \times P} ; \sum_i Q_{i,i+j} = \delta_{i,j} \}
\]

**Algorithm:**

(i) Solve \(D_\lambda(y)\) (SDP program).

(ii) Compute \(\text{supp}(m) = (x_j)_j\) as \(P(e^{2i\pi x_j}) = 0\)

\[P(\xi) \overset{\text{def.}}{=} \xi^{f_c} |\Phi^* p(t) - 1|^2 \quad \text{where} \quad \xi = e^{2i\pi t}
\]
Overview

• Sparse Spikes Super-resolution

• Robust Support Recovery

• Asymptotic Positive Measure Recovery

• Discrete vs. Continuous
Robustness and Support-stability

\[
\min_m \{|m|(\mathbb{T}) ; \Phi m = y\} \quad (\mathcal{P}_0(y))
\]

Low-pass filter \( \text{supp}(\hat{\varphi}) = [-f_c, f_c] \).

When is \( m_0 \) solution of \( \mathcal{P}_0(\Phi m_0) \)?
Robustness and Support-stability

$$\min_{m} \{|m|(\mathbb{T}) ; \Phi m = y\} \quad (\mathcal{P}_0(y))$$

Low-pass filter \(\text{supp}(\hat{\varphi}) = [-f_c, f_c]\).

When is \(m_0\) solution of \(\mathcal{P}_0(\Phi m_0)\) ?

**Theorem:** [Candès, Fernandez G.]

\[\Delta > \frac{1.85}{f_c} \Rightarrow m_0 \text{ solves } \mathcal{P}_0(\Phi m_0).\]
Robustness and Support-stability

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\min_{m} \left\{ |m| (\mathbb{T}) ; \Phi m = y \right\} \quad (\mathcal{P}_0(y))
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\Delta > \frac{1.85}{f_c} \implies m_0 \text{ solves } \mathcal{P}_0(\Phi m_0).
\]

How close to \( m_0 \) are solutions of \( \mathcal{P}_\lambda(\Phi m_0 + w) \)?

- \( \Delta = 0.55/f_c \)
- \( \Delta = 0.45/f_c \)
- \( \Delta = 0.3/f_c \)
- \( \Delta = 0.1/f_c \)
Robustness and Support-stability

\[ \min_m \{|m|(\mathbb{T}) ; \Phi m = y\} \quad (\mathcal{P}_0(y)) \]

Low-pass filter \( \text{supp}(\hat{\varphi}) = [-f_c, f_c] \).

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How close to \( m_0 \) are solutions of \( \mathcal{P}_\lambda(\Phi m_0 + w) \)?

Weighted \( L^2 \) error:

\[ \rightarrow [\text{Candès, Fernandez-G. 2012}] \]

Support localization:

\[ \rightarrow [\text{Fernandez-G.}[\text{de Castro 2012}] \]
Robustness and Support-stability

$$\min_m \left\{ |m|(\mathbb{T}) ; \Phi m = y \right\} \quad (\mathcal{P}_0(y))$$

Low-pass filter \( \text{supp}(\hat{\phi}) = [-f_c, f_c] \).

When is \( m_0 \) solution of \( \mathcal{P}_0(\Phi m_0) \) ?

**Theorem:** [Candès, Fernandez G.]

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**Open problems:** Exact support recovery? General kernels?
Dual Certificates

**Noiseless recovery:**  
\[
\min_{\Phi(m) = \Phi(m_0)} |m|(T) \quad (\mathcal{P}_0)
\]
**Dual Certificates**

Noiseless recovery:
\[
\min_{\Phi(m) = \Phi(m_0)} |m|(\mathbb{T}) \quad (P_0)
\]

Proposition:
\[
m_0 \text{ solution of } (P_0) \iff \exists \eta \in D(m_0)
\]

**Dual certificates:**
\[
D(m_0) \overset{\text{def.}}{=} \text{Im}(\Phi^*) \cap \partial|m_0|(\mathbb{T})
\]

\[
\partial|m_{a,x}|(\mathbb{T}) = \{\eta \in C(\mathbb{T}) \mid \|\eta\|_\infty \leq 1, \forall i, \eta(x_i) = \text{sign}(a_i)\}
\]

\[
\Delta = 1/f_c
\]
**Dual Certificates**

**Noiseless recovery:**
\[
\min_{\Phi(m) = \Phi(m_0)} |m|(\mathbb{T}) \quad (P_0)
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\]

\[
\Delta = \frac{1}{f_c} \quad \eta
\]

\[
\Delta = 0.6/f_c \quad \eta
\]
**Dual Certificates**

**Noiseless recovery:**
\[
\min_{\Phi(m) = \Phi(m_0)} |m|(\mathbb{T}) \quad (\mathcal{P}_0)
\]

**Proposition:**
\[m_0 \text{ solution of } (\mathcal{P}_0) \iff \exists \eta \in \mathcal{D}(m_0)
\]

**Dual certificates:**
\[\mathcal{D}(m_0) \overset{\text{def.}}{=} \text{Im}(\Phi^*) \cap \partial|m_0|(\mathbb{T})
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\]

**Non-degenerate certificate:**
\[\eta \in \bar{\mathcal{D}}(m_{a,x}) \iff \forall s \notin x, |\eta(s)| < 1 \quad \forall s \in x, \eta''(s) \neq 0
\]
Minimal-norm certificate: $\eta_0 = \arg\min_{\eta=\Phi^*p} \|p\| \quad \text{s.t.} \quad \eta \in D(m_0)$
\textbf{Support Stability}

\textit{Minimal-norm certificate:} \( \eta_0 = \arg\min_{\eta = \Phi^* p} \| p \| \quad \text{s.t.} \quad \eta \in \mathcal{D}(m_0) \)

\textbf{Theorem:} If \( \eta_0 \in \bar{\mathcal{D}}(m_0) \) for \( m_0 = m_{a_0, x_0} \), then for \( (\| w \| / \lambda, \lambda) = O(1) \),
the solution of \( \mathcal{P}_\lambda(y) \) for \( y = \Phi(m_0) + w \) is
\[
\sum_{i=1}^{N} a_i^* \delta x_i^* \quad \text{where} \quad \| (x_0, a_0) - (x^*, a^*) \| = O(\| w \|).
\]

[Duval, Peyré 2014]
Support Stability

**Minimal-norm certificate:** \( \eta_0 = \arg\min_{\eta = \Phi^* p} \| p \| \) s.t. \( \eta \in \mathcal{D}(m_0) \)

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Support Stability

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\sum_{i=1}^{N} a_i^* \delta x_i^* \quad \text{where} \quad \| (x_0, a_0) - (x^*, a^*) \| = O(\|w\|).
\]

Noiseless \( w = 0 \).

[Duval, Peyré 2014]
Vanishing Derivative Pre-Certificate

$$\eta_V = \arg\min_{\eta = \Phi^* p} \|p\| \quad \text{s.t.} \quad \forall \ i \begin{cases} \eta(x_{0,i}) = \text{sign}(a_{0,i}), \\ \eta'(x_{0,i}) = 0. \end{cases}$$
Vanishing Derivative Pre-Certificate
\[ \eta_V = \operatorname*{arg\,min}_{\eta=\Phi^* p} \|p\| \quad \text{s.t.} \quad \forall i \left\{ \begin{array}{l} \eta(x_0,i) = \operatorname{sign}(a_{0,i}), \\ \eta'(x_0,i) = 0. \end{array} \right. \]

**Proposition:** \[ \eta_V = \Phi^* A_{x_0}^+ (\operatorname{sign}(a_0); 0) \]
where \[ A_x(b) = \sum_i b_i^1 \varphi(x_i, \cdot) + b_i^2 \varphi'(x_i, \cdot) \]
Vanishing Derivative (Pre-)Certificate

where

\[ A_x(b) = \sum_i b_i^1 \varphi(x_i, \cdot) + b_i^2 \varphi'(x_i, \cdot) \]

**Proposition:** \( \eta_V = \Phi^* A_{x_0}^+ (\text{sign}(a_0); 0) \)

where \( A_x(b) = \sum_i b_i^1 \varphi(x_i, \cdot) + b_i^2 \varphi'(x_i, \cdot) \)

**Theorem:** \( \eta_V \in \overline{D}(m_0) \implies \eta_0 = \eta_V \)

\[ \eta_V = \text{argmin}_{\eta = \Phi^* p} \|p\| \quad \text{s.t.} \quad \forall i \left\{ \begin{array}{l} \eta(x_0, i) = \text{sign}(a_0, i), \\ \eta'(x_0, i) = 0. \end{array} \right. \]

\[ \eta_0 = \text{argmin}_{\eta = \Phi^* p, \eta_0} \|p\| \quad \text{s.t.} \quad \forall i \left\{ \begin{array}{l} \eta(x_0, i) = \text{sign}(a_0, i), \\ \eta'(x_0, i) = 0. \end{array} \right. \]
When is $\eta_V$ Non-degenerate?

Input measure: $m_0 = m_{a_0, \Delta x_0}, \quad \Delta \to 0$
When is $\eta_V$ Non-degenerate?

**Input measure:** $m_0 = m_{a_0, \Delta x_0}, \quad \Delta \to 0$

**Theorem:** [Tang, Recht, 2013]

$\exists C, (\Delta > C \sigma) \implies (\eta_V \text{ is non degenerate})$

Valid for:

- $\varphi(x) = e^{-x^2/\sigma^2}$
- $\varphi(x) = (1 + (x/\sigma)^2)^{-1}$

$\varphi(x) = e^{-x^2/\sigma^2}$
Overview

- Sparse Spikes Super-resolution

- Robust Support Recovery

- Asymptotic Positive Measure Recovery

- Discrete vs. Continuous
Recovery of Positive Measures

\[ m_0 = m_{a_0, x_0} \quad \text{where} \quad a_0 \in \mathbb{R}^N_+ \]

**Theorem:** let \[ \Phi m = \left( \int e^{-2i\pi kt} dm(t) \right)_{k=-f_c}^{f_c} \] and \[ \eta_S(t) = 1 - \rho \prod_{i=1}^{N} \sin(\pi(t - x_i))^2 \]

for \( N \leq f_c \) and \( \rho \) small enough, \( \eta_S \in \bar{D}(m_0) \).

\[ \rightarrow m_0 \] is recovered when there is no noise.

[de Castro et al. 2011]
Recovery of Positive Measures

\[ m_0 = m_{a_0, x_0} \quad \text{where} \quad a_0 \in \mathbb{R}_+^N \]

Theorem: let \( \Phi m = (\int e^{-2i\pi kt} dm(t))_{k=-f_c}^{f_c} \) and
\[ \eta_S(t) = 1 - \rho \prod_{i=1}^N \sin(\pi(t-x_i))^2 \]
for \( N \leq f_c \) and \( \rho \) small enough, \( \eta_S \in \bar{D}(m_0) \).

\[ \rightarrow m_0 \text{ is recovered when there is no noise.} \]
\[ \rightarrow \text{behavior as } x_0 \to 0? \]
\( m_0 = m_{a_0, x_0} \) where \( a_0 \in \mathbb{R}_+^N \)

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\( \rightarrow m_0 \) is recovered when there is no noise.

\( \rightarrow \) behavior as \( x_0 \to 0 \) ?

[Morgenshtern, Candès, 2015] discrete \( \ell^1 \) robustness.

[Demanet, Nguyen, 2015] discrete \( \ell^0 \) robustness.
Recovery of Positive Measures

\[ m_0 = m_{a_0, x_0} \quad \text{where} \quad a_0 \in \mathbb{R}^N_+ \]

**Theorem:** let \( \Phi m = (\int e^{-2i\pi kt} \, dm(t))_{k= -f_c}^{f_c} \) and

\[ \eta_S(t) = 1 - \rho \prod_{i=1}^{N} \sin(\pi(t - x_i))^2 \]

for \( N \leq f_c \) and \( \rho \) small enough, \( \eta_S \in \bar{D}(m_0) \).

\[ \rightarrow m_0 \text{ is recovered when there is no noise.} \]

\[ \rightarrow \text{behavior as } x_0 \to 0 ? \]

[de Castro et al. 2011]

[Morgenshtern, Candès, 2015] discrete \( \ell^1 \) robustness.

[Demianet, Nguyen, 2015] discrete \( \ell^0 \) robustness.

\[ \rightarrow \text{noise robustness of support recovery?} \]
Comparison of Certificates

$\eta_S$

$\eta_V$
Asymptotic of Vanishing Certificate

\[ m_0 = m_{a_0, \Delta x_0} \quad \text{where} \quad \Delta \to 0 \]

**Vanishing Derivative pre-certificate:**

\[ \eta V \overset{\text{def.}}{=} \arg\min_{\eta = \Phi^* p} \|p\| \]

\[ \text{s.t.} \quad \forall \, i, \quad \begin{cases} 
\eta(\Delta x_{0,i}) = 1, \\
\eta'(\Delta x_{0,i}) = 0.
\end{cases} \]
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**Asymptotic pre-certificate:**
\[
\eta_W \overset{\text{def.}}{=} \text{argmin} \|p\| \\
\eta = \Phi^*p \\
\text{s.t.} \quad \begin{cases} 
\eta(0) = 1, \\
\eta'(0) = \ldots = \eta^{(2N-1)}(0) = 0.
\end{cases}
\]
Asymptotic Certificate

\[
(2N - 1)\text{-Non degenerate:}
\]

\[
\eta_W \in \overline{D}_N
\]

\[
\iff \quad \forall x \neq 0, |\eta_W(x)| < 1
\]

\[
\eta_W^{(2N)}(0) \neq 0
\]
Asymptotic Certificate

(2N - 1)-Non degenerate:

\( \eta_W \in \bar{D}_N \)

\( \iff \{ \forall x \neq 0, |\eta_W(x)| < 1 \)

\( \eta_W^{(2N)}(0) \neq 0 \)

Lemma:

If \( \eta_W \in \bar{D}_N \), \( \exists \Delta_0 > 0 \),

\( \forall \Delta < \Delta_0, \eta_V \in \bar{D}(m_{\Delta x_0, a_0}) \)

\( \rightarrow \eta_W \) govern stability as \( \Delta \to 0 \).
Theorem: If $\eta_W \in \bar{D}_N$, letting $m_0 = m_{a_0, \Delta x_0}$, then
for $\left( \frac{w}{\lambda}, \frac{w}{\Delta^{2N-1}}, \frac{\lambda}{\Delta^{2N-1}} \right) = O(1)$

the solution of $P_\lambda(y)$ for $y = \Phi(m_0) + w$ is

\[ \sum_{i=1}^{N} a_i^* \delta \Delta x_i^* \text{ where } \| (x_0, a_0) - (x^*, a^*) \| = O \left( \frac{\|w\| + \lambda}{\Delta^{2N-1}} \right) \]
**Theorem:** If $\eta_W \in \bar{D}_N$, letting $m_0 = m_{a_0, \Delta x_0}$, then for $(\frac{w}{\lambda}, \frac{w}{\Delta^{2N-1}}, \frac{\lambda}{\Delta^{2N-1}}) = O(1)$ the solution of $\mathcal{P}_\lambda(y)$ for $y = \Phi(m_0) + w$ is

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where

$$\| (x_0, a_0) - (x^*, a^*) \| = O \left( \frac{\|w\| + \lambda}{\Delta^{2N-1}} \right)$$

**Asymptotic Robustness**

$$y = \Phi m_{a_0,tx_0} + w$$

<table>
<thead>
<tr>
<th>Noise: $w = \lambda w_0$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regularization: $\lambda = \lambda_0 \Delta^\alpha$</td>
</tr>
</tbody>
</table>

$\alpha = 2N - 1$

$\alpha < 2N - 1$
| Proposition: one has $\eta_W^{(2N)}(0) < 0.$ | → “locally” non-degenerate. |
When is \( \eta_W \) Non-degenerate?

**Proposition:** one has \( \eta_W^{(2N)}(0) < 0 \). → “locally” non-degenerate.
Overview

- **Sparse Spikes Super-resolution**
- **Robust Support Recovery**
- **Asymptotic Positive Measure Recovery**
- **Discrete vs. Continuous**
Measures:
\[
\min_m \frac{1}{2} \| \Phi(m) - y \|^2 + \lambda |m|_{(\mathbb{T})}
\]

On a grid \( z \):
\[
\min_{a \in \mathbb{R}^N} \frac{1}{2} \| \Phi(m_{a,z}) - y \|^2 + \lambda \|a\|_1
\]
Discrete vs. Continuous Recovery

Measures:
\[
\min_m \frac{1}{2} \| \Phi(m) - y \|^2 + \lambda |m|(\mathbb{T})
\]

On a grid \( z \):
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\min_{a \in \mathbb{R}^N} \frac{1}{2} \| \Phi(m_{a,z}) - y \|^2 + \lambda \|a\|_1
\]

If \( m_0 = m_{a_0,z} \), then for fine grids \( z \), \( \text{supp}(a^*) \neq \text{supp}(a_0) \).

\[\rightarrow \text{Discrete support is not stable!}\]
Discrete Minimal Norm Certificate

$$\min_{a \in \mathbb{R}^N} \frac{1}{2} \| \Phi_z(a) - y \|^2 + \lambda \| a \|_1 \quad \text{where} \quad \Phi_z(a) = \Phi(m_{a,z})$$
\[
\min_{a \in \mathbb{R}^N} \frac{1}{2} \| \Phi_z(a) - y \|^2 + \lambda \| a \|_1 \quad \text{where} \quad \Phi_z(a) = \Phi(m_{a,z})
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Discrete certificate:
\[
I = \text{supp}(a_0) \quad \bar{D}_d(m_0) = \left\{ \eta \in \text{Im}(\Phi^*): \begin{array}{c}
|\eta(z_{Ic})| < 1 \\
\eta(z_I) = \text{sign}(a_0, I)
\end{array} \right\}
\]
Discrete Minimal Norm Certificate

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\min_{a \in \mathbb{R}^N} \frac{1}{2} \| \Phi_z(a) - y \|^2 + \lambda \| a \|_1 \quad \text{where} \quad \Phi_z(a) = \Phi(m_{a,z})
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Fuch’s pre-certificate:
\[
\eta_F = \Phi^+_{z_I} \text{sign}(a_{0,I})
\]
**Discrete Minimal Norm Certificate**

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\min_{a \in \mathbb{R}^N} \frac{1}{2} \| \Phi_z(a) - y \|^2 + \lambda \| a \|_1 \quad \text{where} \quad \Phi_z(a) = \Phi(m_a, z)
\]

**Discrete certificate:**

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I = \text{supp}(a_0) \quad \bar{D}_d(m_0) = \left\{ \eta \in \text{Im}(\Phi^*) : \begin{array}{l}
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**Fuch’s pre-certificate:**

\[
\eta_F = \Phi_{z_I}^+, \text{sign}(a_0, I)
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**Theorem:** \( \text{supp}(a_0) \) is stable at low noise iff \( \eta_F \in \bar{D}_d(m_0) \).
Discrete Minimal Norm Certificate

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\min_{a \in \mathbb{R}^N} \frac{1}{2} \| \Phi_z(a) - y \|^2 + \lambda \| a \|_1 \quad \text{where} \quad \Phi_z(a) = \Phi(m_{a,z})
\]

Discrete certificate: \( I = \text{supp}(a_0) \) \( \bar{D}_d(m_0) = \left\{ \eta \in \text{Im}(\Phi^*) : \begin{aligned} |\eta(z_{I_c})| &< 1 \\ \eta(z_I) &= \text{sign}(a_0,I) \end{aligned} \right\} \)

Fuch’s pre-certificate: \( \eta_F = \Phi_{z_I}^+ \ast \text{sign}(a_0,I) \)

Theorem: \( \text{supp}(a_0) \) is stable at low noise iff \( \eta_F \in \bar{D}_d(m_0) \).

\rightarrow \eta_F \notin \bar{D}_d(m_0) \) for fine grids!
**Theorem:** if \( \eta_0 \in \tilde{D}(m_0) \),

then for fine grids and for \((\|w\|/\lambda, \lambda) = O(1)\),

\[
\text{supp}(a^*) \subset \text{supp}(a_0) \oplus \{-1, 0, 1\}
\]
Conclusion

Deconvolution of measures:

\[ L^2 \text{ errors are not well-suited.} \]

Weak-* convergence.

Optimal transport distance.

Exact support estimation.

\[ \Delta = 0.45 / f_c \]

\[ \Delta = 0.55 / f_c \]
Deconvolution of measures:

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\[ \Delta = \frac{0.45}{f_\epsilon} \]

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Low-noise behavior:

→ dictated by $\eta_0$.

→ checkable via $\eta_V$.

→ asymptotic via $\eta_W$. 
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Lasso on discrete grids: similar $\eta_0$-analysis applies.
  → Relate discrete and continuous recoveries.
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$\rightarrow$ checkable via $\eta_V$.

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Lasso on discrete grids: similar $\eta_0$-analysis applies.

$\rightarrow$ Relate discrete and continuous recoveries.

Open problem: other regularizations (e.g. piecewise constant)?