Exact Support Recovery for Sparse Spikes Deconvolution

Gabriel Peyré

Joint work with
Vincent Duval & Quentin Denoyelle

www.numerical-tours.com

Sparse Deconvolution

Neural spikes (1D)

$$y = \varphi \star m_0 + w$$
m_0 is "sparse"

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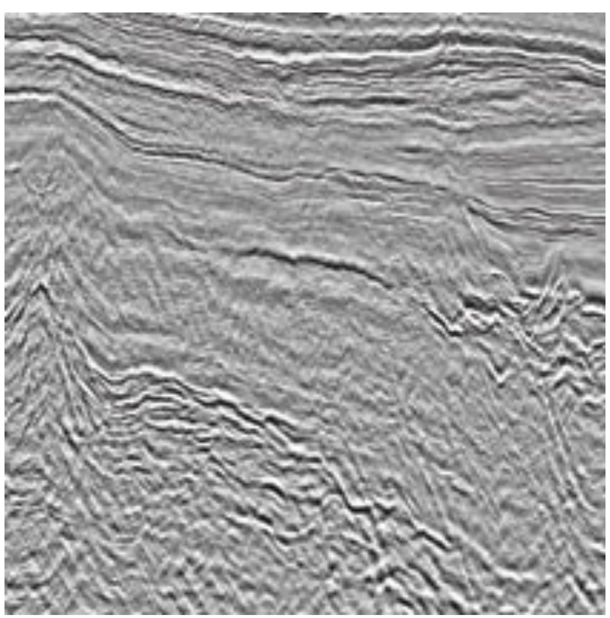

Seismic imaging (1.5D)

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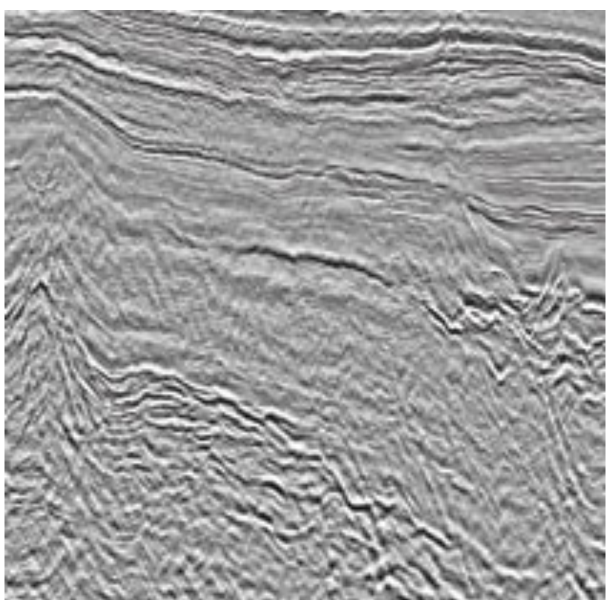

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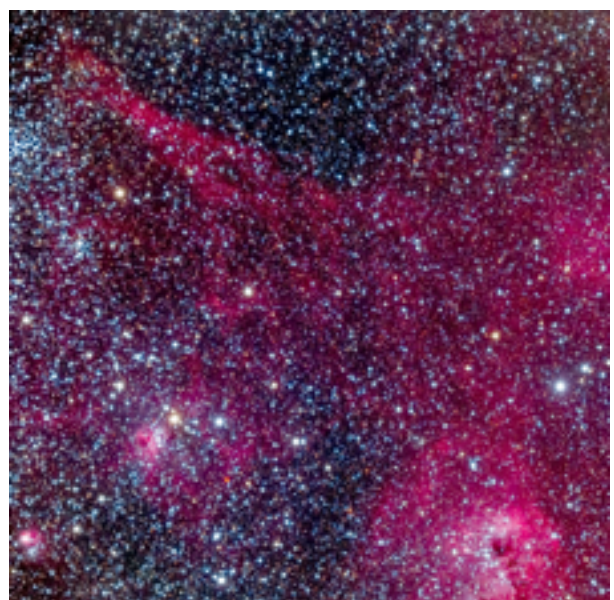

Astrophysics (2D)

Sparse Deconvolution

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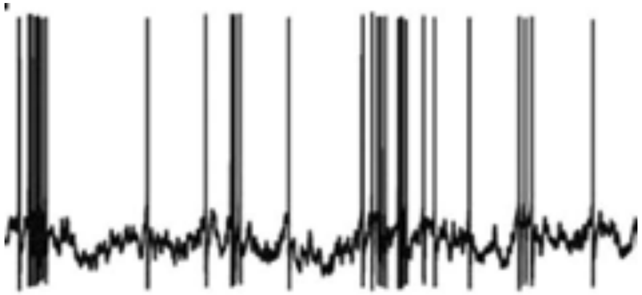
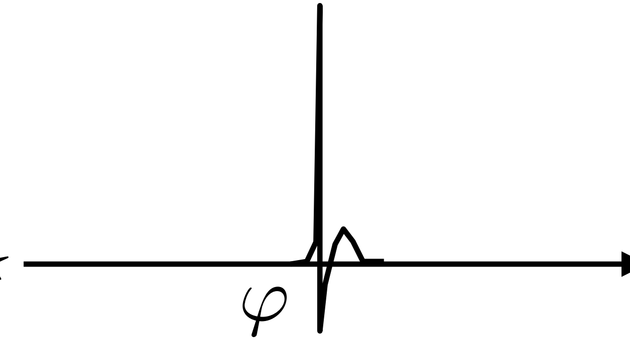


y $=$

m_0 $\star$



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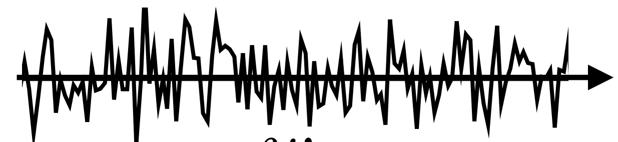

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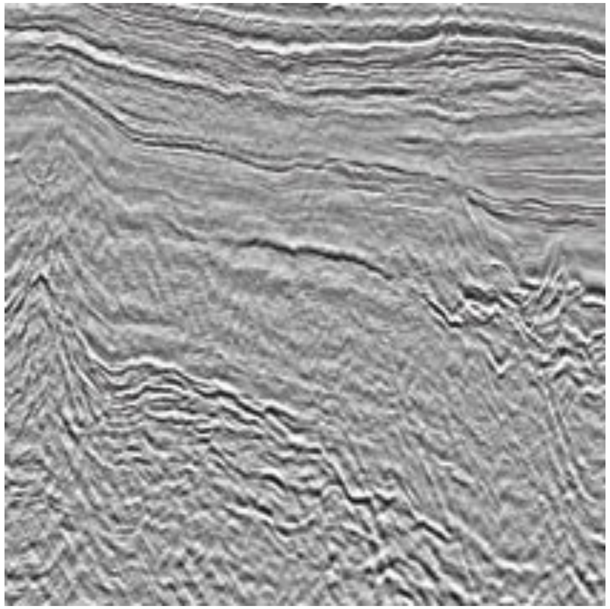


Seismic imaging (1.5D)

Astrophysics (2D)

Presented results extend to nD problems

Overview

- Sparse Spikes Super-resolution

- Robust Support Recovery

- Asymptotic Positive Measure Recovery

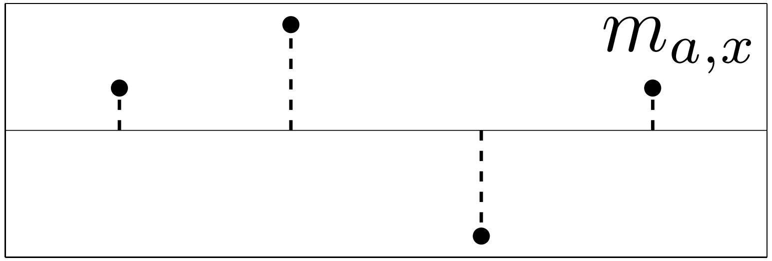
- Discrete vs. Continuous

Deconvolution of Measures

Radon measure m on $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^d$.

Discrete measure:

$$m_{a,x} = \sum_{i=1}^{N} a_i \delta_{x_i}, \; a \in \mathbb{R}^N, x \in \mathbb{T}^N$$



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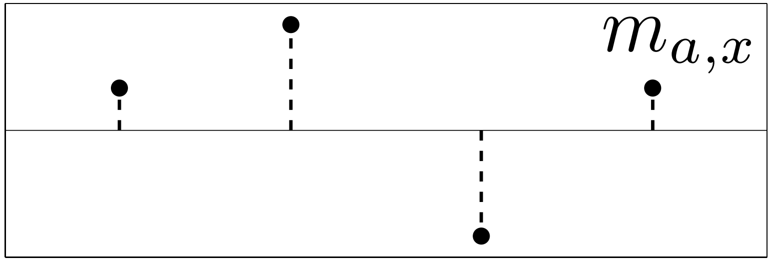

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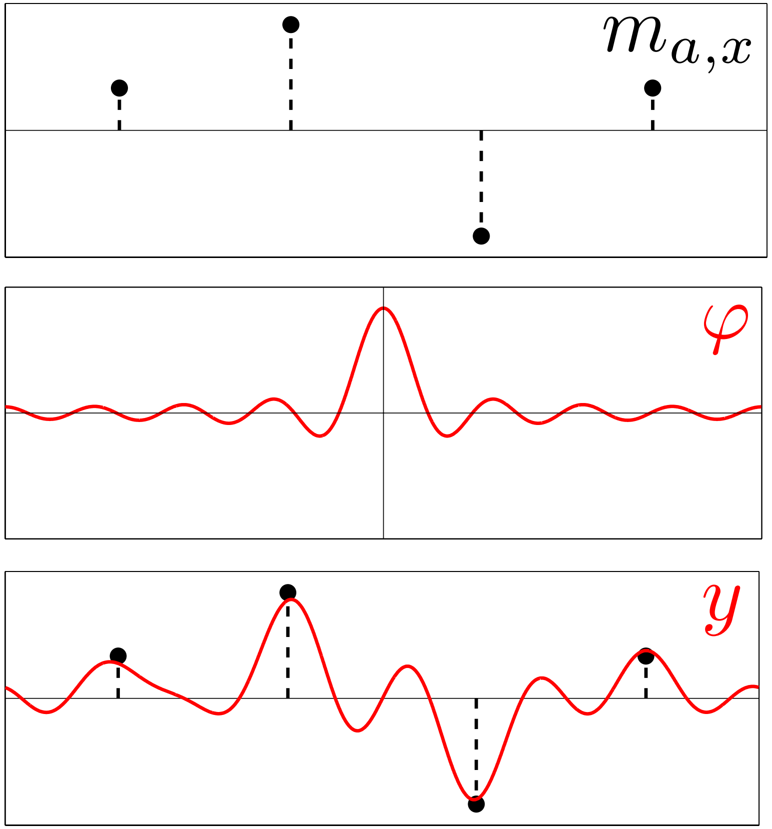
Example: 1-D ($d = 1$) convolution

$$\varphi(x, t) = \varphi(x - t)$$



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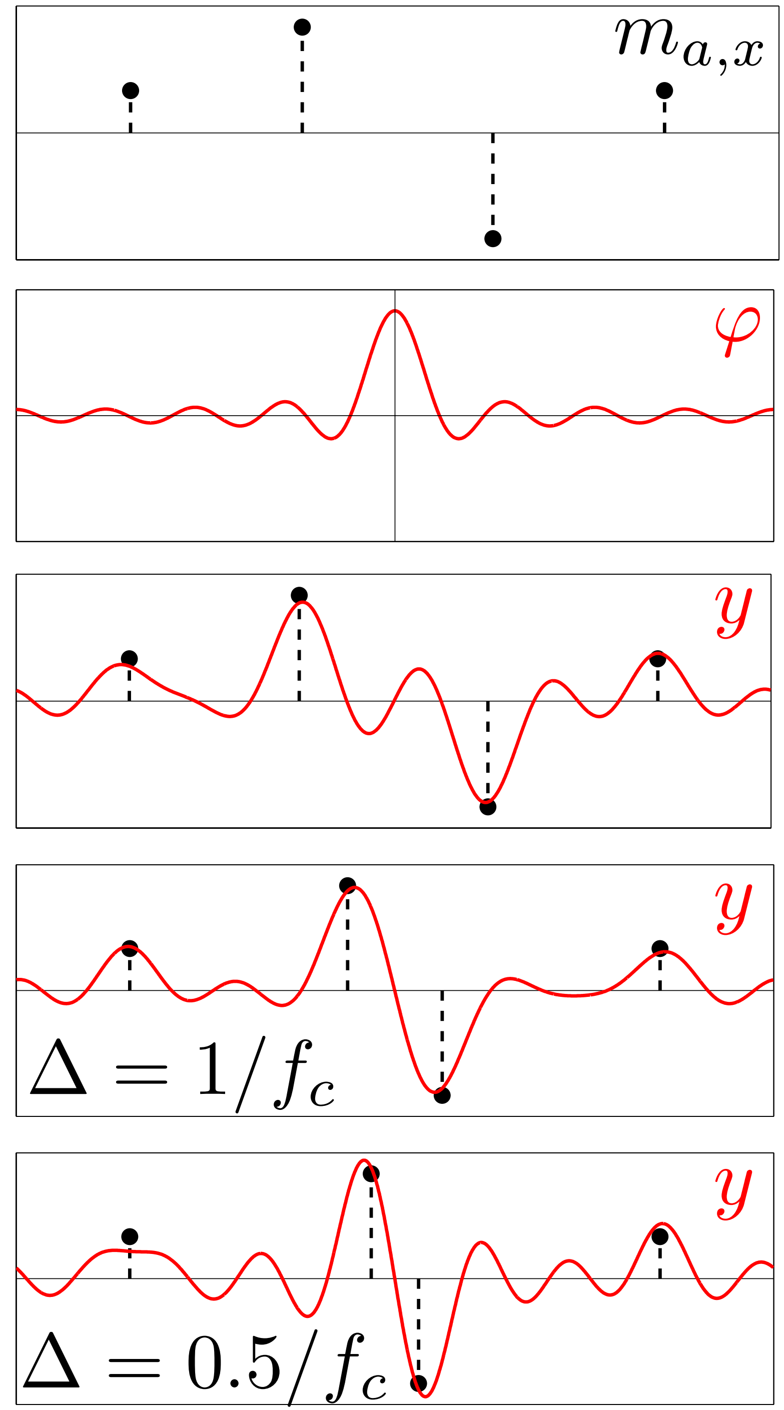
Minimum separation:

$$\Delta = \min_{i \neq j} |x_i - x_j|$$

$\rightarrow$ Signal-dependent recovery criteria.



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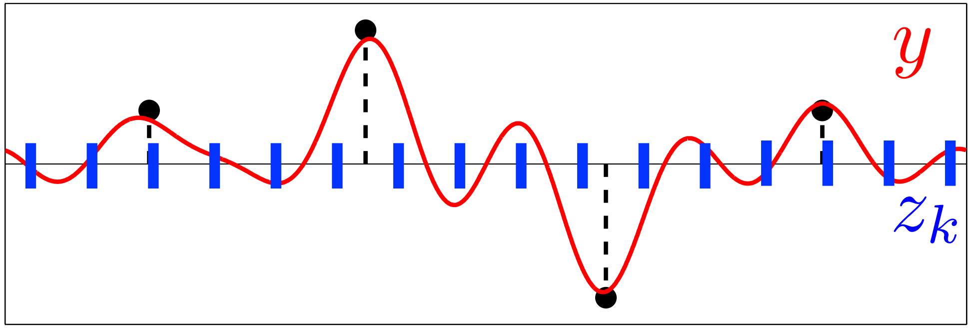

Discrete ℓ^1 regularization:

Computation grid $z = (z_k)_{k=1}^K$.

Basis-pursuit / Lasso:

$$\min_{a \in \mathbb{R}^N} \frac{1}{2}\|y - \Phi(m_{a,z})\|^2 + \lambda \|a\|_1$$

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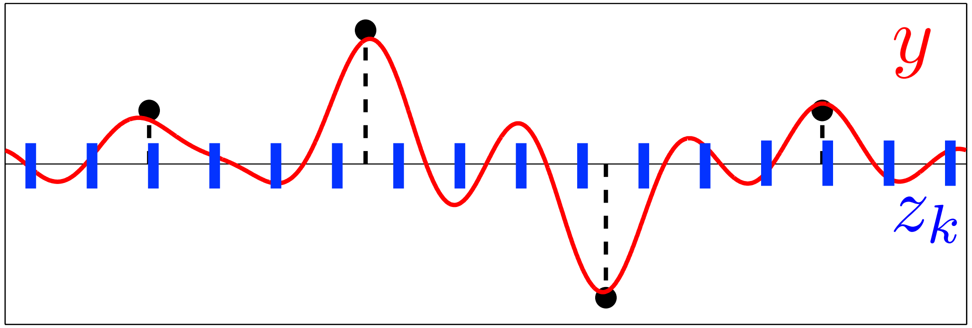

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Grid-free regularization: total variation of measures:

$$|m|(\mathbb{T}) = \sup \left\{ \textstyle\int \eta \mathrm{d}m \ : \ \eta \in C(\mathbb{T}), \|\eta\|_\infty \leqslant 1 \right\}$$

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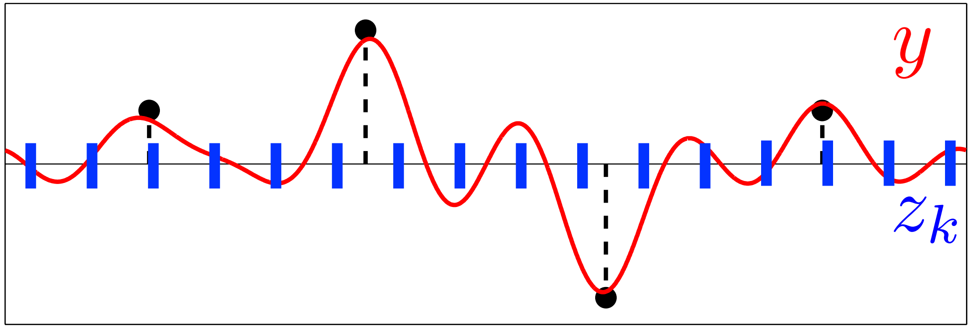

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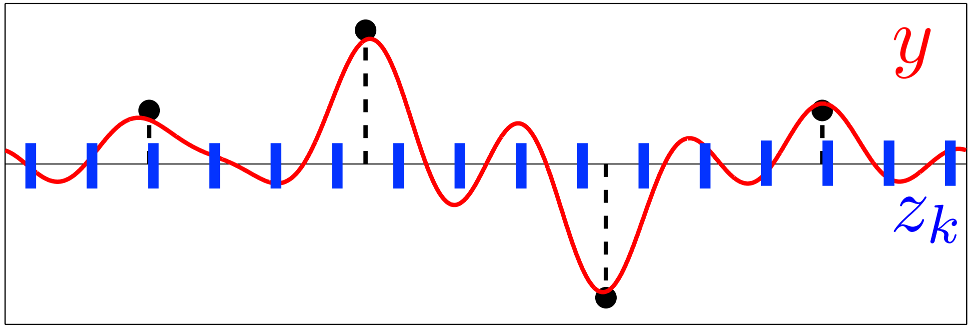

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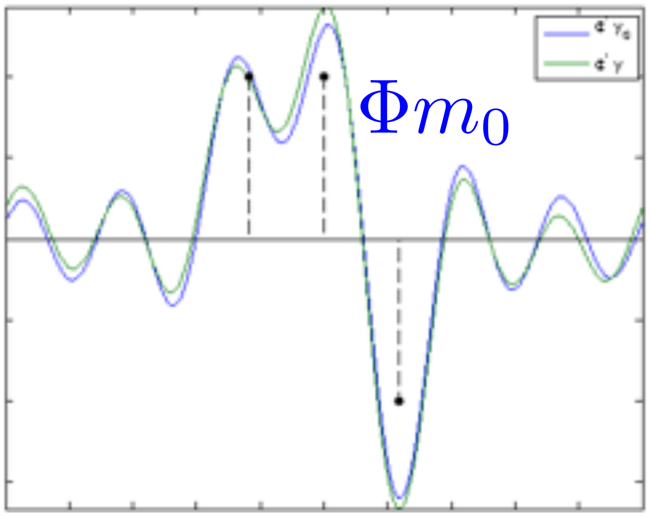
$\longrightarrow$ Algorithms: [Bredies, Pikkarainen, 2010] (proximal-based)
[Candès, Fernandez-G. 2012] (root finding)

Dual Minimization

Primal program:

$$\min_{m} \frac{1}{2}\|\Phi(m) - y\|^2 + \lambda |m|(\mathbb{T})$$



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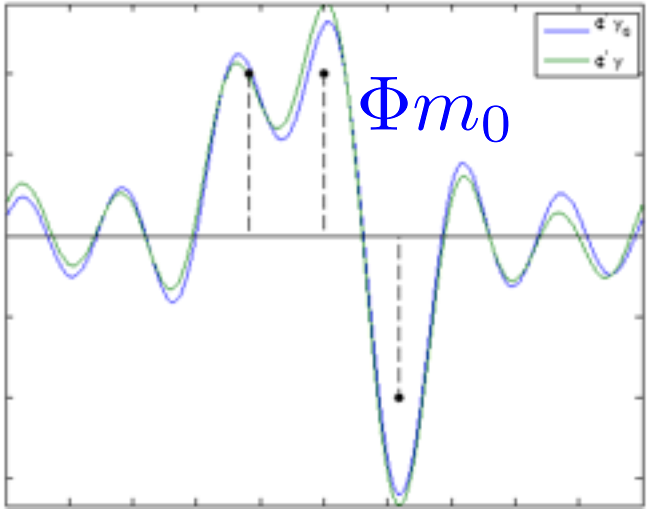
$$\min_{\substack{\|\eta\|_\infty \leqslant 1 \\ \eta = \Phi^* p}} \|y/\lambda - p\|$$



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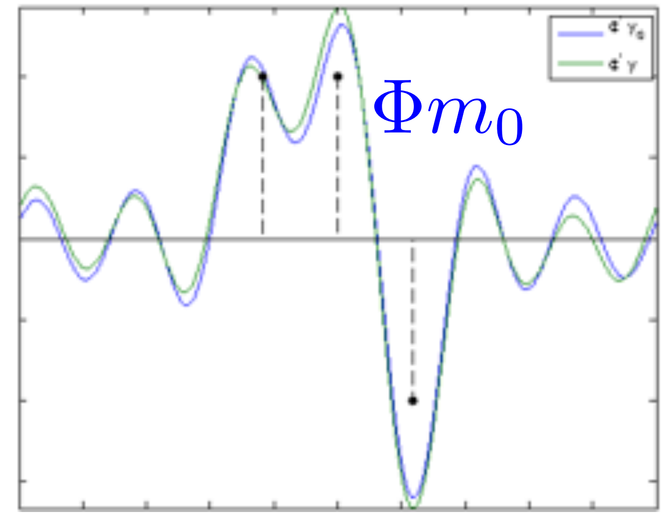
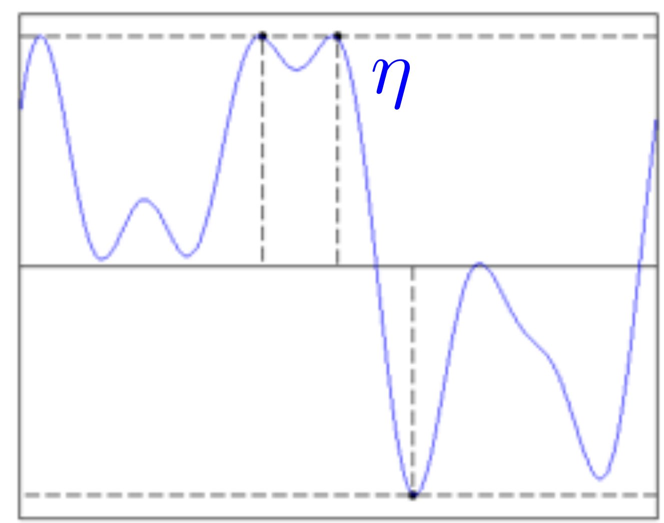
Proposition: [primal-dual relations]

$$\mathrm{supp}(m) \subset \{t \; ; \; |\eta(t)| = 1\} \quad (1)$$

$$\eta = \lambda^{-1}\Phi^*(y - \Phi m) \quad (2)$$





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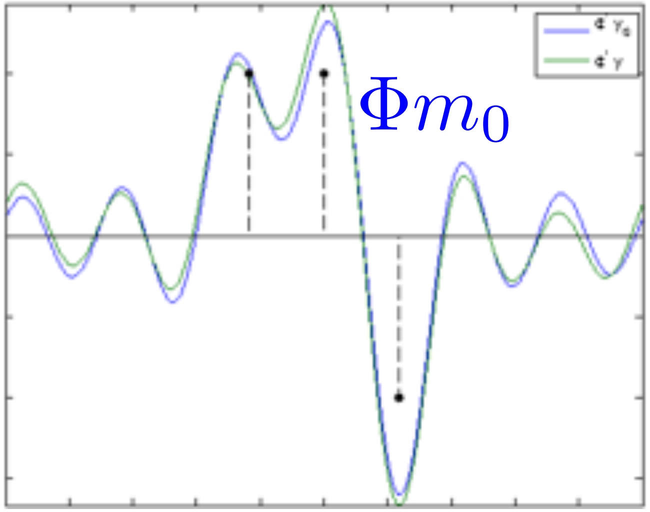

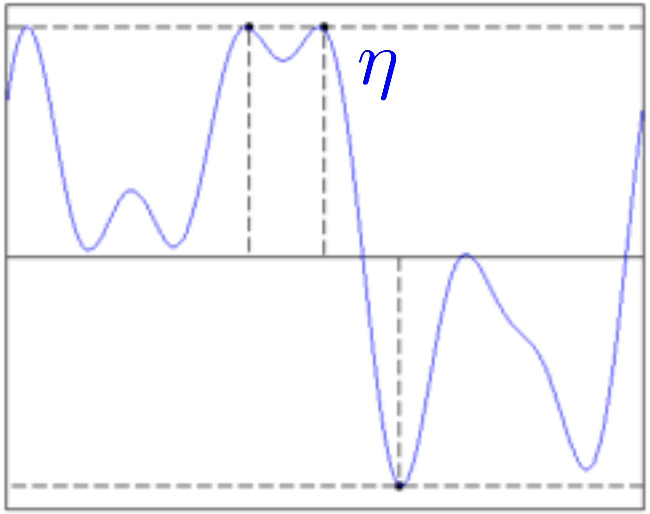

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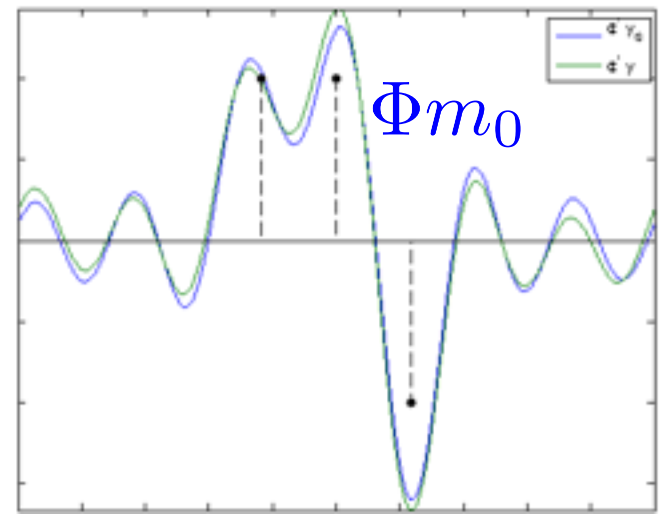
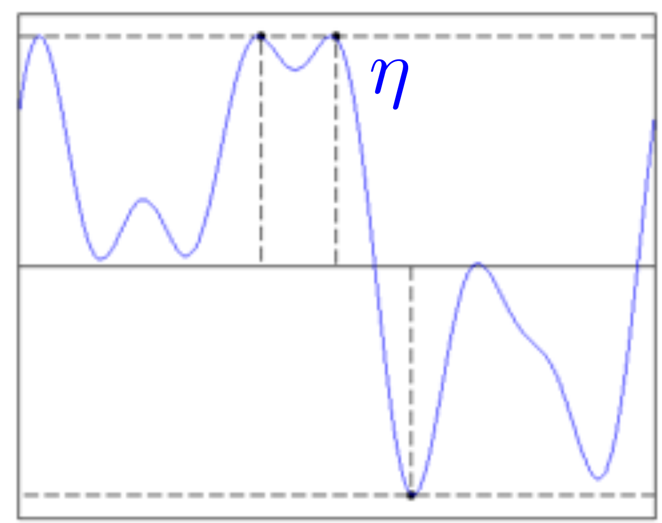
$$a = \Phi_x^+ y - \lambda(\Phi_x^* \Phi_x)^{-1}\eta(x)$$



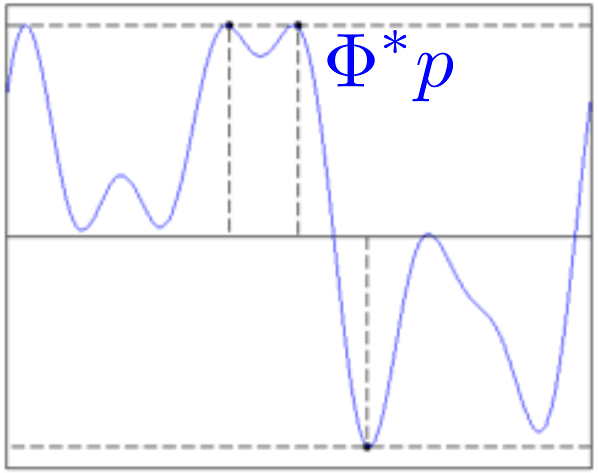


SDP Dual Resolution and Root Finding

Low frequency measurements:

$$(\Phi^* p)(t) = \sum_{|k| \leqslant f_c} p_k e^{2\mathrm{i}\pi k t}$$

in 1D



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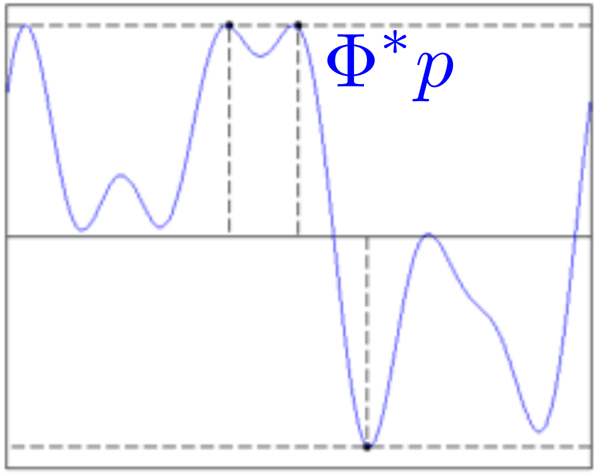

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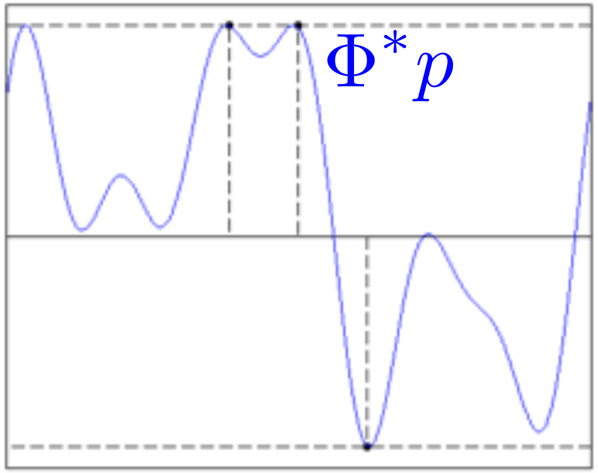

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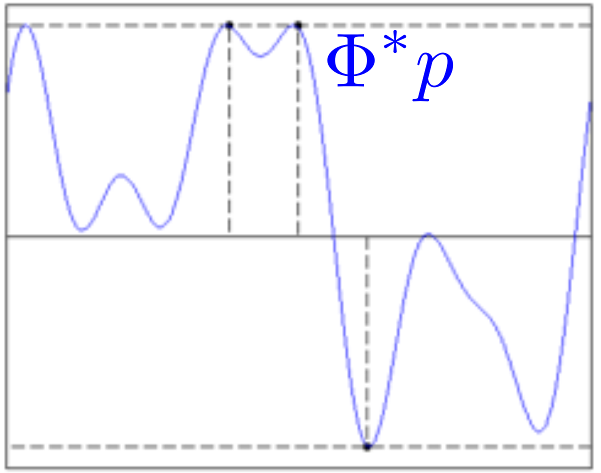

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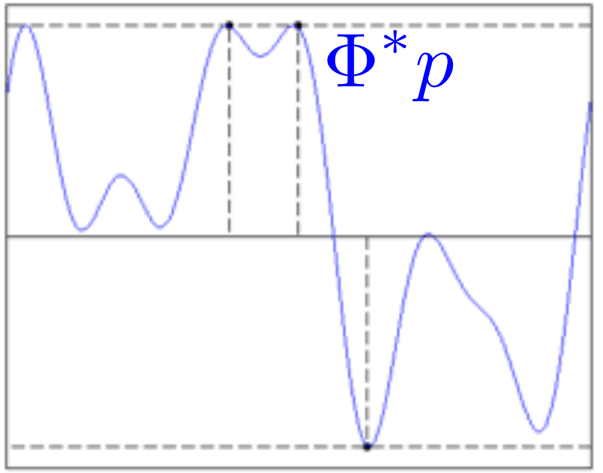

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(ii) Compute $\text{supp}(m) = (x_j)_j$ as $P(e^{2i\pi x_j}) = 0$

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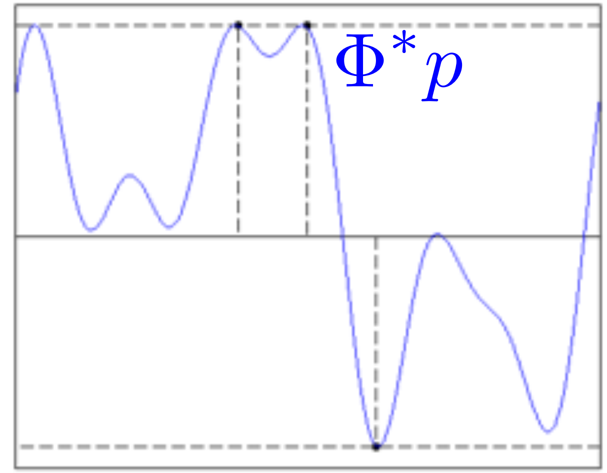

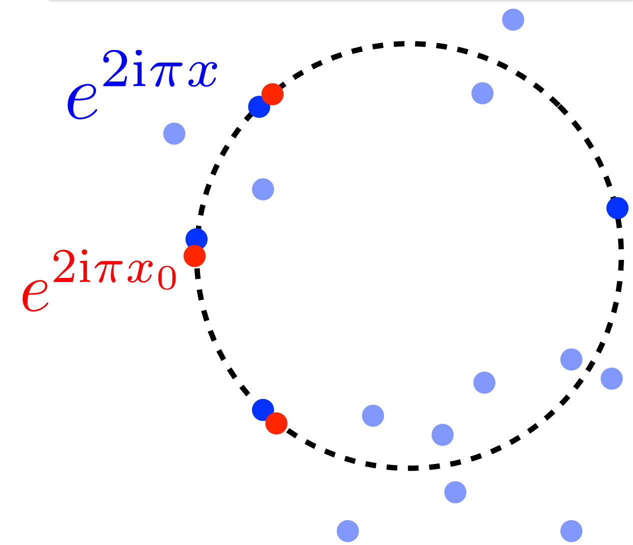

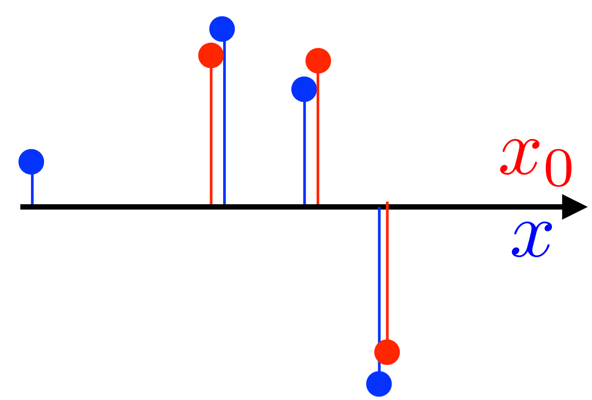

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- **Robust Support Recovery**
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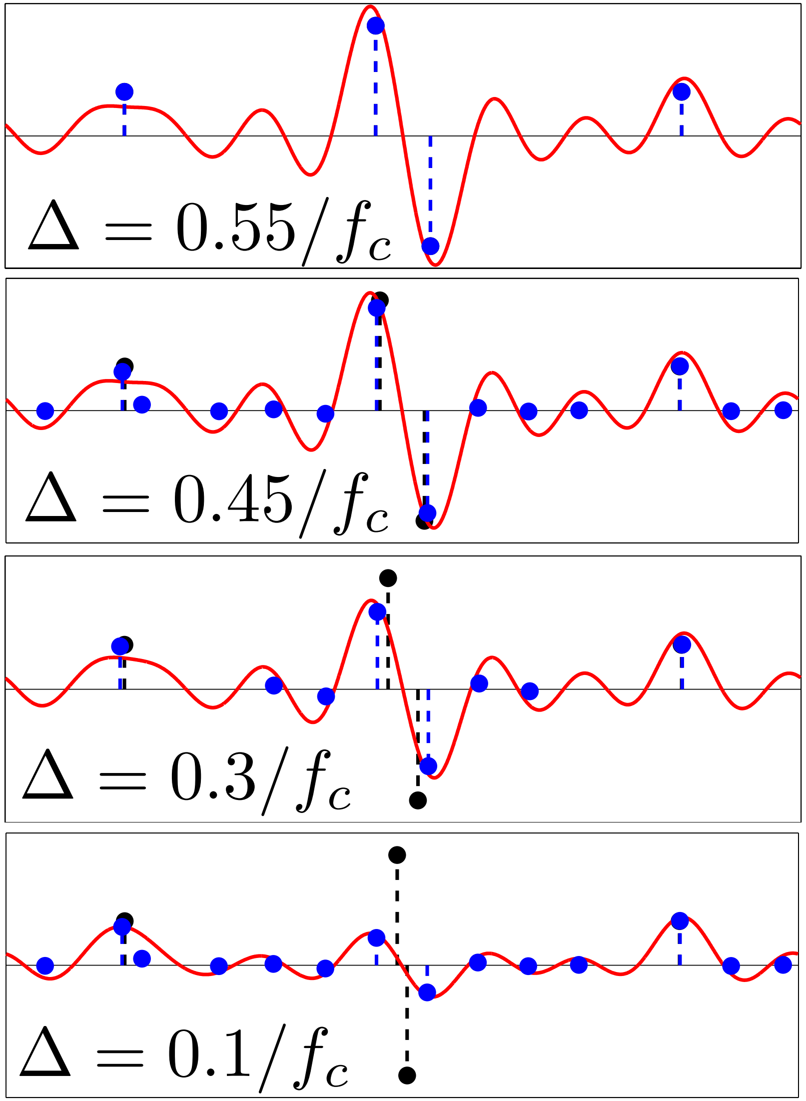
- Discrete vs. Continuous

Robustness and Support-stability

$$\min_m \; \{|m|(\mathbb{T}) \;;\; \Phi m = y\} \quad (\mathcal{P}_0(y))$$

Low-pass filter $\text{supp}(\hat{\varphi}) = [-f_c, f_c]$.

When is m_0 solution of $\mathcal{P}_0(\Phi m_0)$?



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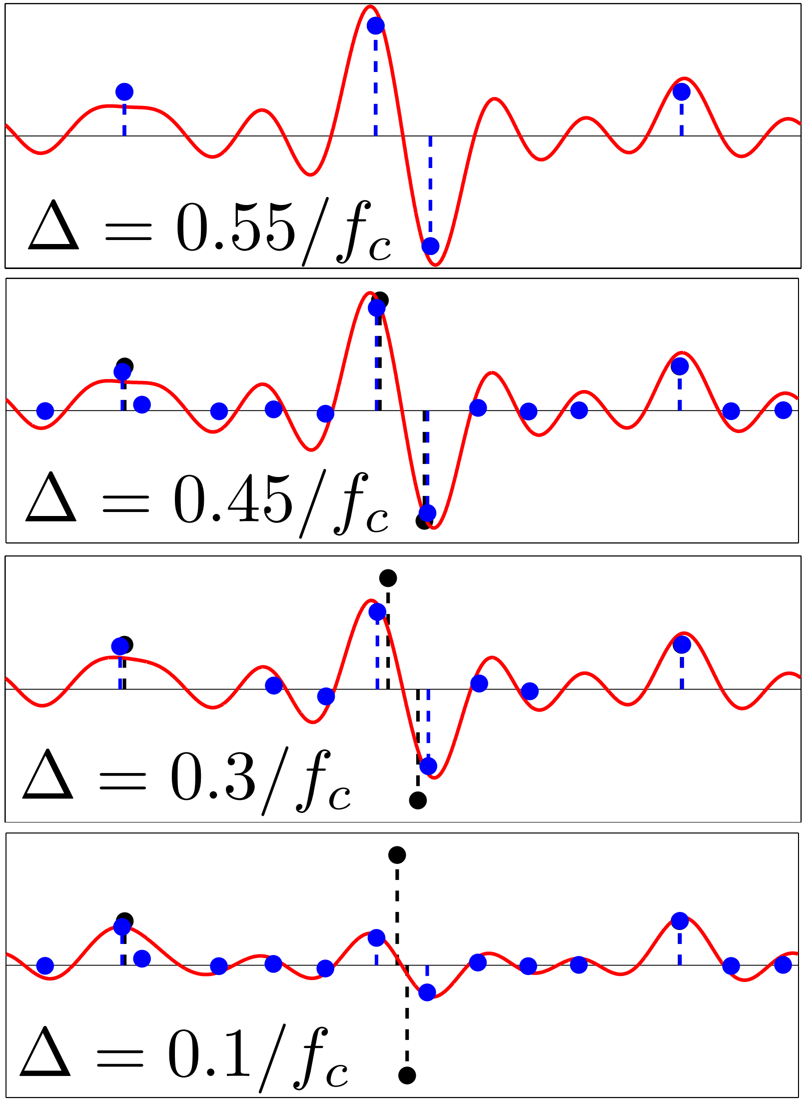
Theorem: [Candès, Fenandez G.]

$\Delta > \frac{1.85}{f_c} \Rightarrow m_0$ solves $\mathcal{P}_0(\Phi m_0)$.



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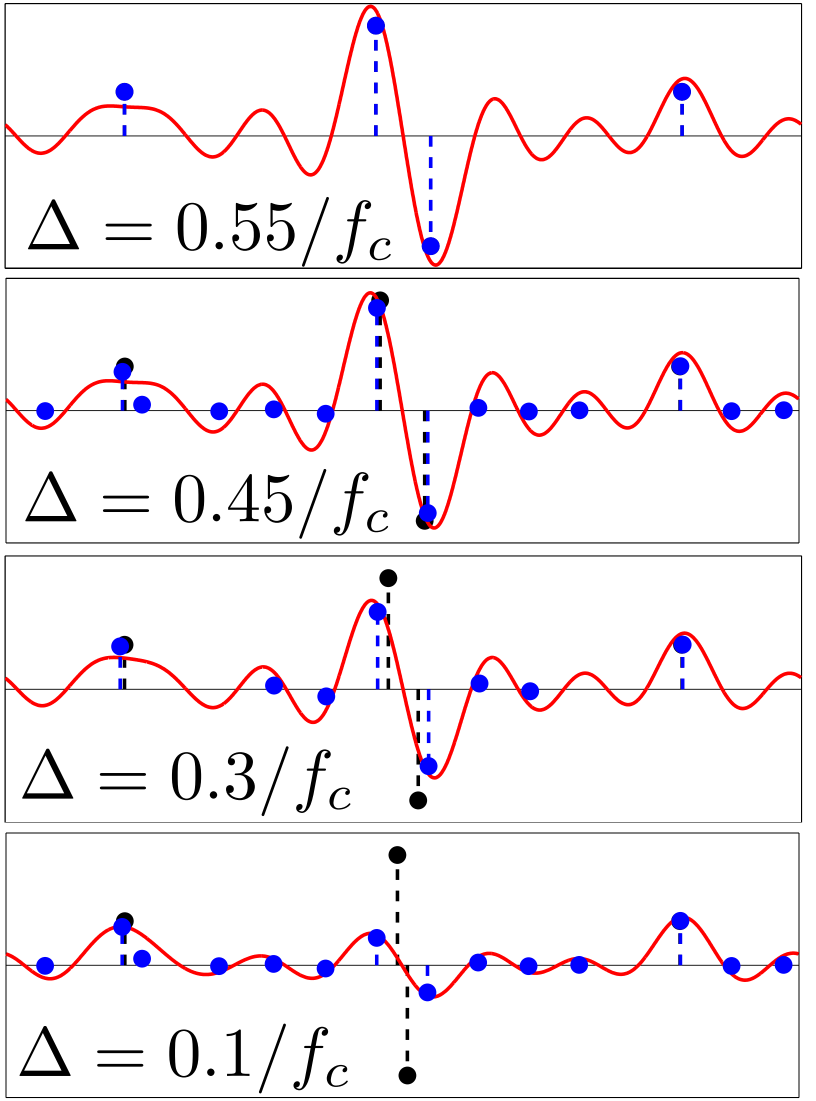

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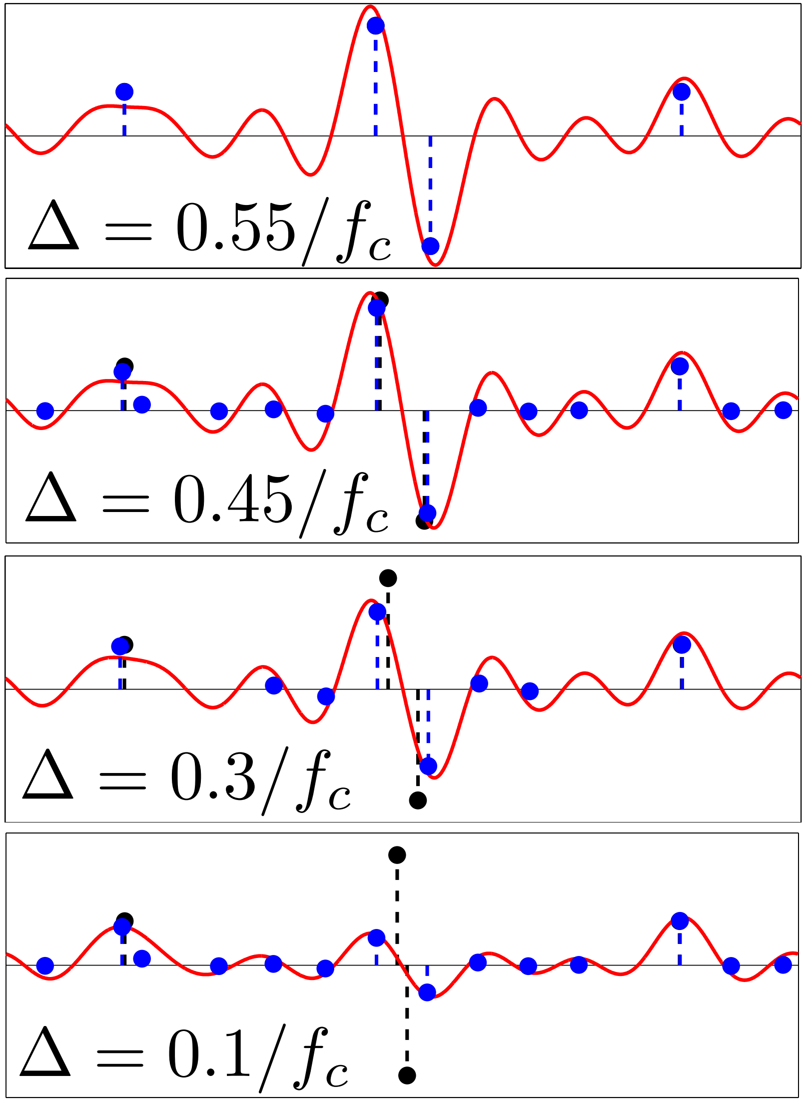
Weighted L^2 error:
$\longrightarrow$ [Candès, Fernandez-G. 2012]

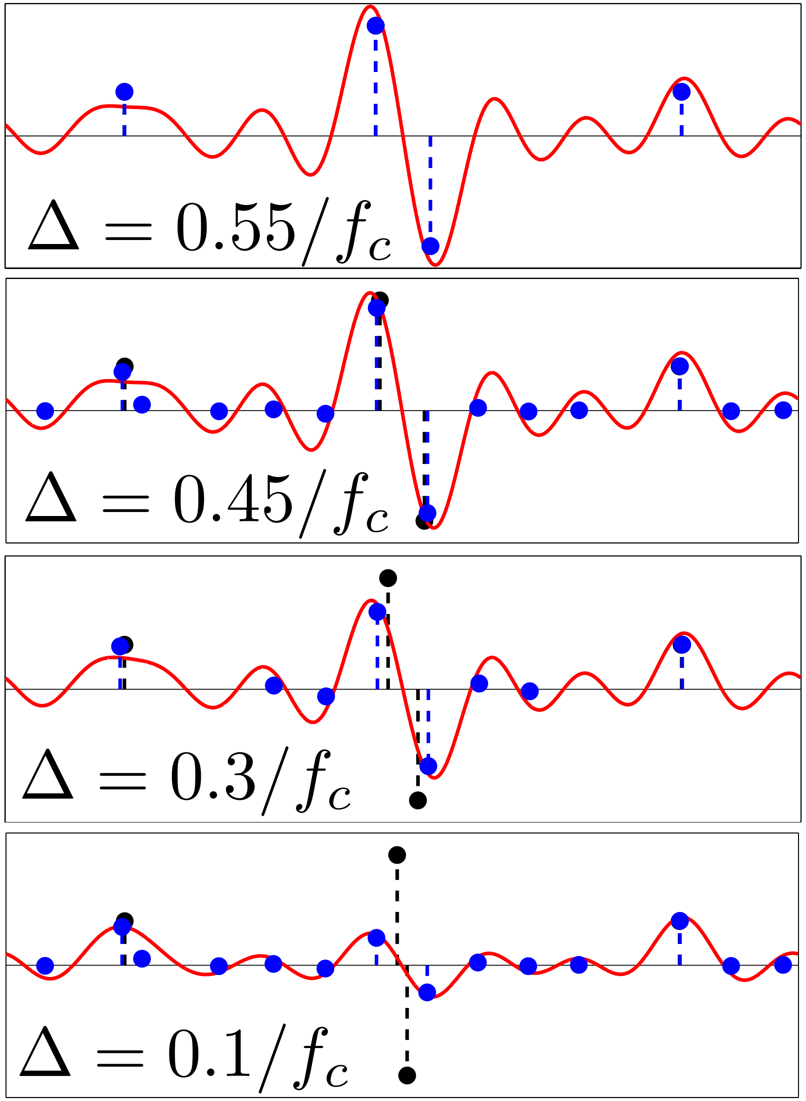
Support localization:
$\longrightarrow$ [Fernandez-G.][de Castro 2012]



Robustness and Support-stability

$$\min_m \{|m|(\mathbb{T})\ ;\ \Phi m = y\} \quad (\mathcal{P}_0(y))$$

Low-pass filter $\text{supp}(\hat\varphi) = [-f_c, f_c]$.



When is m_0 solution of $\mathcal{P}_0(\Phi m_0)$?

Theorem: [Candès, Fenandez G.]
$\Delta > \frac{1.85}{f_c} \Rightarrow m_0$ solves $\mathcal{P}_0(\Phi m_0)$.

How close to m_0 are solutions of $\mathcal{P}_\lambda(\Phi m_0 + w)$?

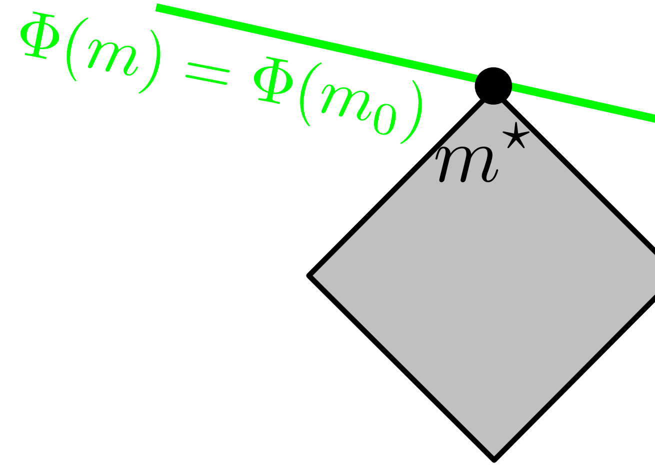
Weighted L^2 error:
$\longrightarrow$ [Candès, Fernandez-G. 2012]

Support localization:
$\longrightarrow$ [Fernandez-G.][de Castro 2012]

Open problems: Exact support recovery? General kernels?

Dual Certificates

Noiseless recovery: $\min_{\Phi(m)=\Phi(m_0)} |m|(\mathbb{T}) \quad (\mathcal{P}_0)$



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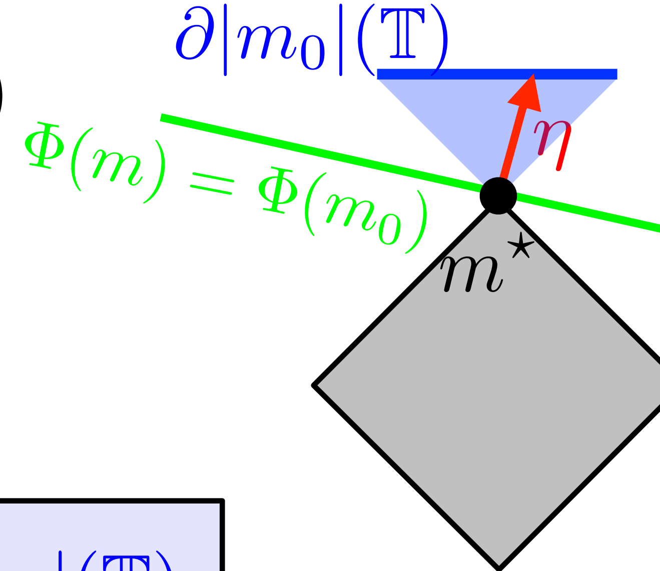
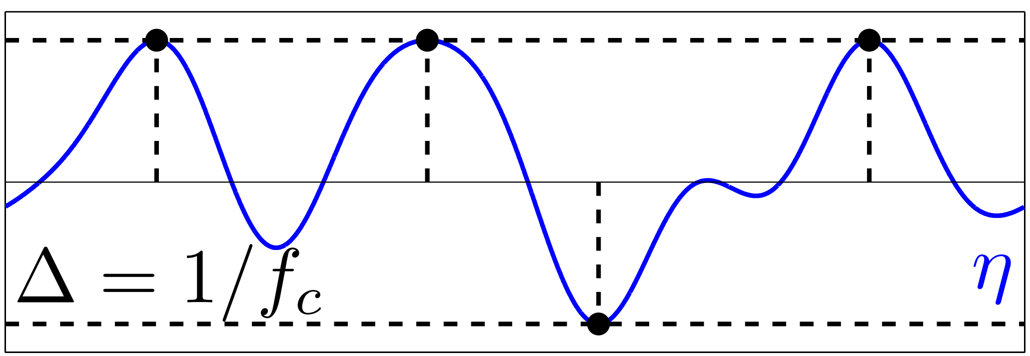


Proposition:

m_0 solution of $(\mathcal{P}_0) \iff \exists\, \eta \in \mathcal{D}(m_0)$

Dual certificates: $\mathcal{D}(m_0) \overset{\text{def.}}{=} \text{Im}(\Phi^*) \cap \partial|m_0|(\mathbb{T})$

$$\partial|m_{a,x}|(\mathbb{T}) = \{\eta \in C(\mathbb{T}) \; ; \; \|\eta\|_\infty \leqslant 1, \; \forall\, i, \;\; \eta(x_i) = \text{sign}(a_i)\}$$



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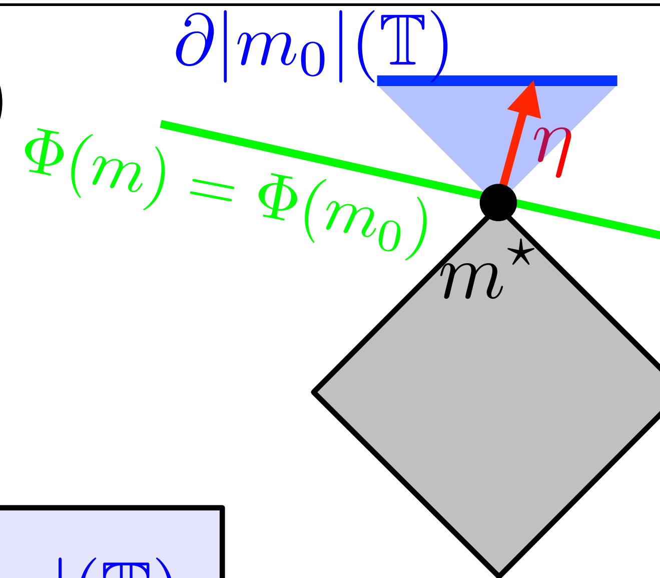

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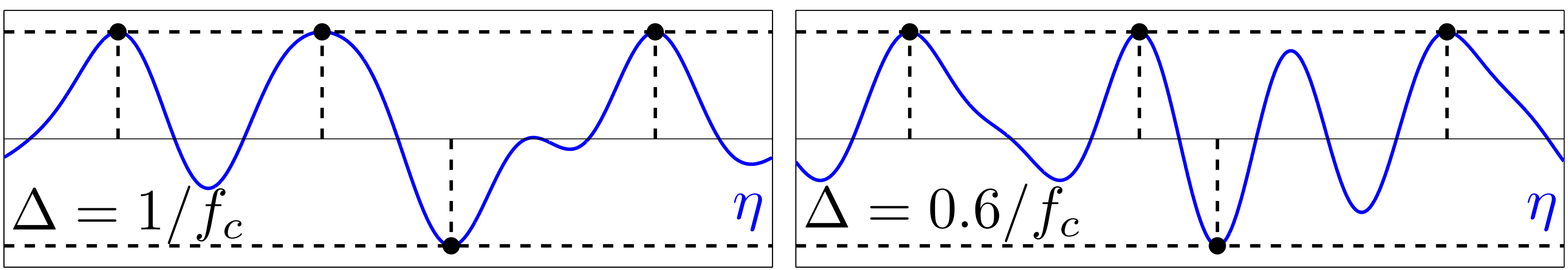

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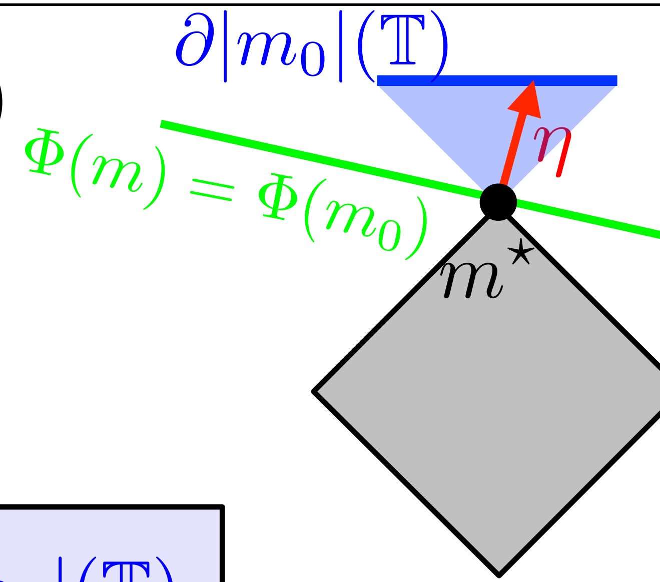

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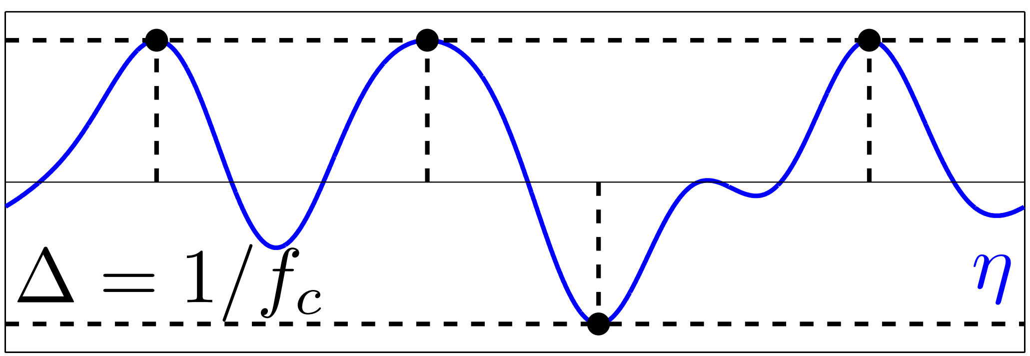

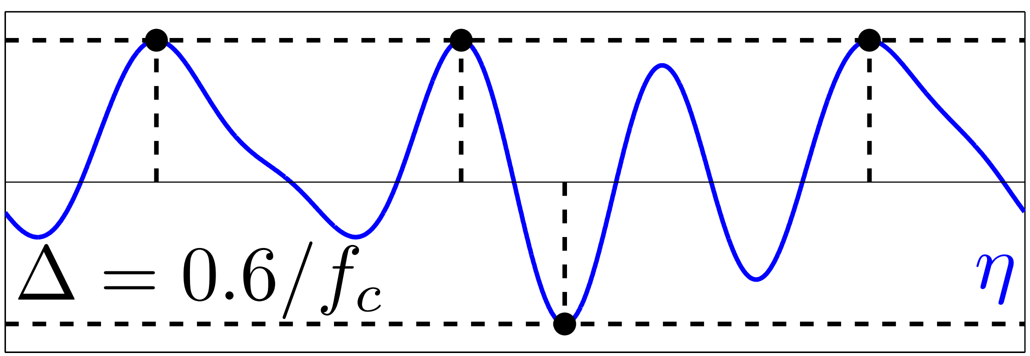

Non-degenerate certificate: $\eta \in \bar{\mathcal{D}}(m_{a,x})$

$\iff \quad \forall\, s \notin x,\ |\eta(s)| < 1 \qquad \forall\, s \in x,\ \eta''(s) \neq 0$

Support Stability

Minimal-norm certificate: $\eta_0 = \underset{\eta=\Phi^* p}{\text{argmin}} \|p\| \quad \text{s.t.} \quad \eta \in \mathcal{D}(m_0)$

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[Duval, Peyré 2014]

Theorem: If $\eta_0 \in \bar{\mathcal{D}}(m_0)$ for $m_0 = m_{a_0,x_0}$, then
for $(\|w\|/\lambda, \lambda) = O(1)$,
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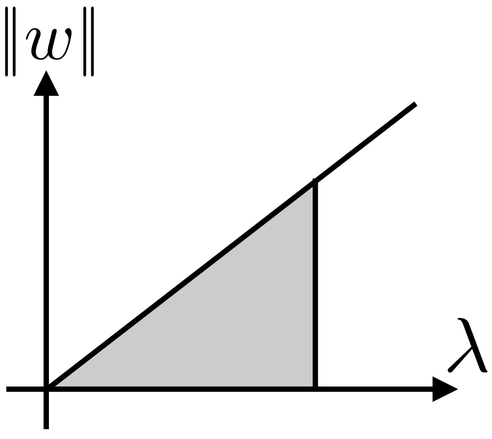

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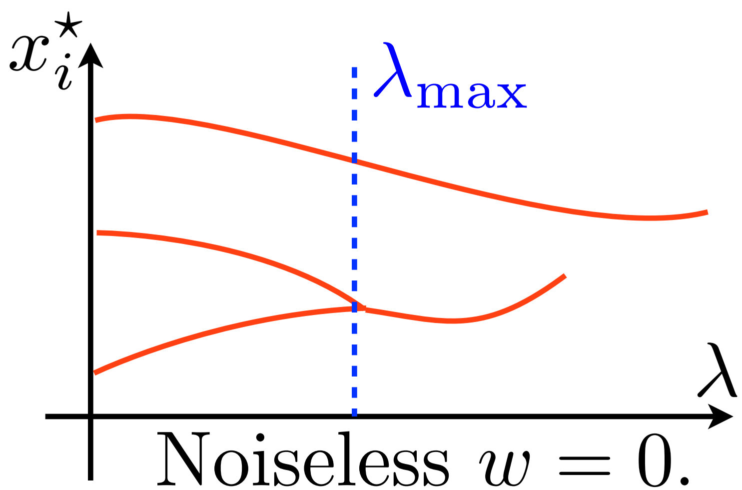

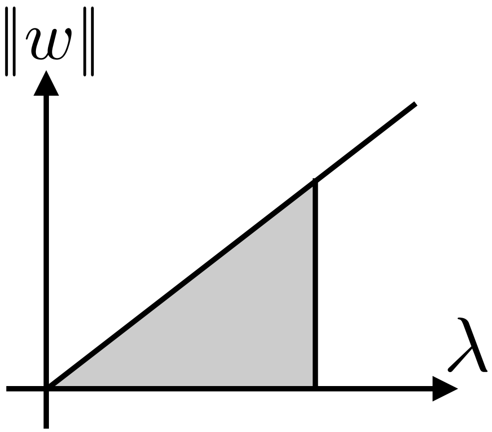

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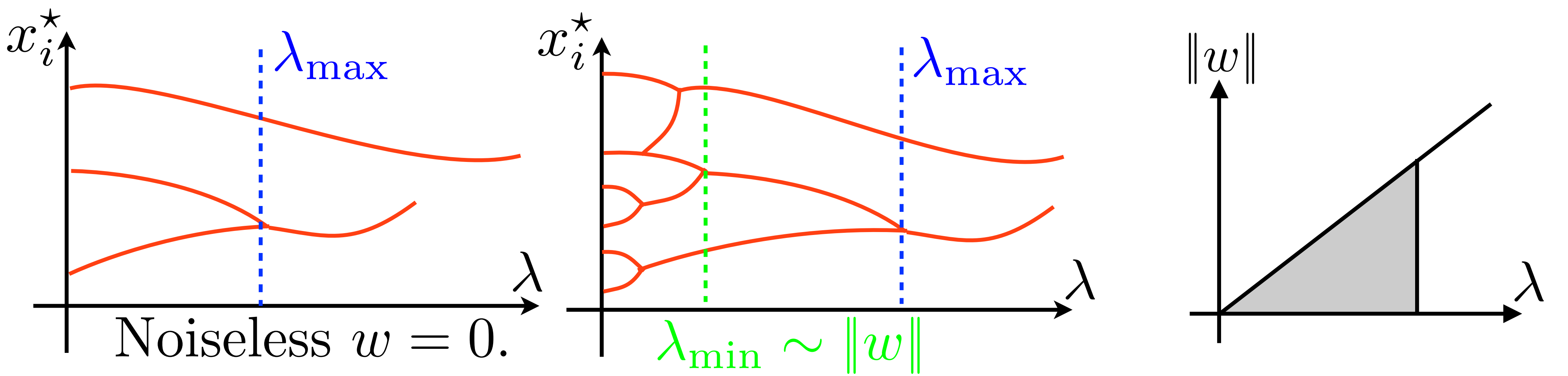
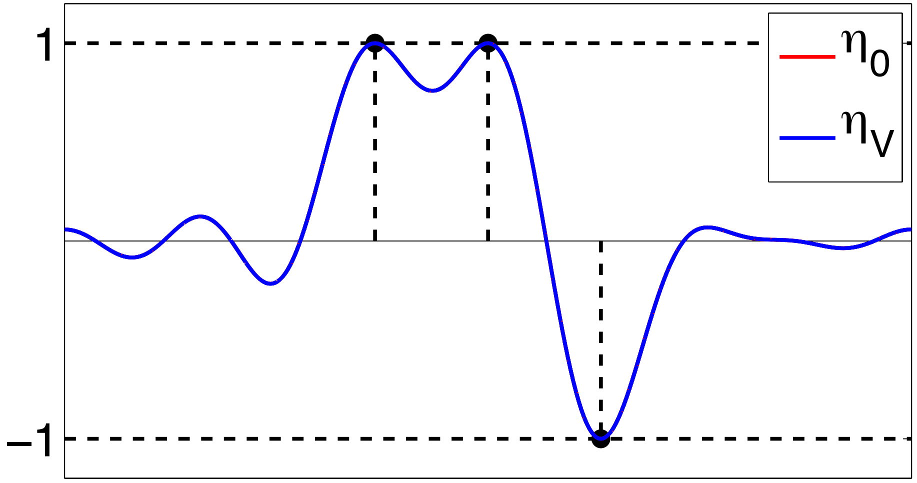


Vanishing Derivative (Pre-)Certificate

Vanishing Derivative Pre-Certificate

$$\eta_V = \underset{\eta=\Phi^* p}{\text{argmin}} \|p\| \quad \text{s.t.} \quad \forall\, i \begin{cases} \eta(x_{0,i}) = \text{sign}(a_{0,i}), \\ \eta'(x_{0,i}) = 0. \end{cases}$$



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Proposition: $\eta_V = \Phi^* A_{x_0}^{+,*}(\text{sign}(a_0); 0)$

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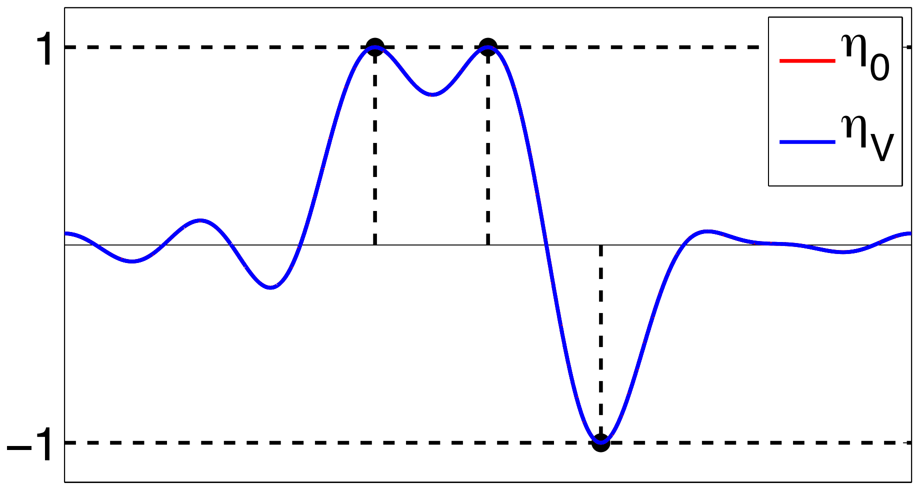

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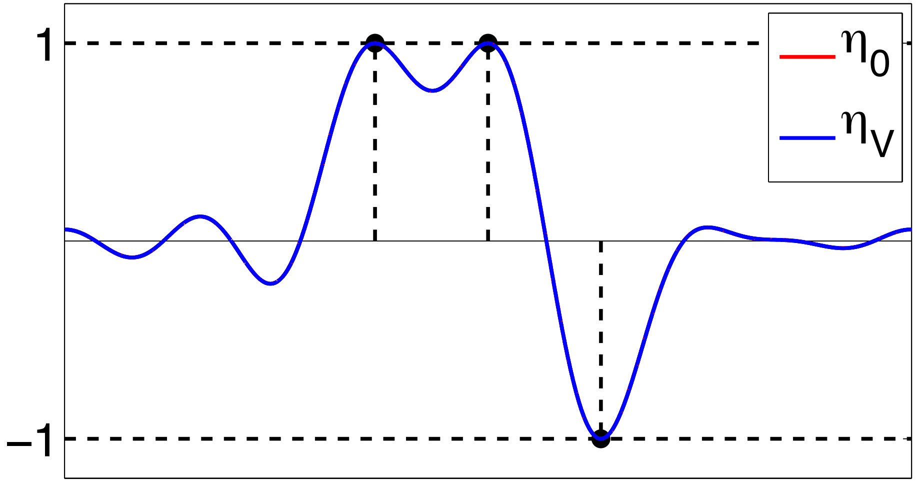

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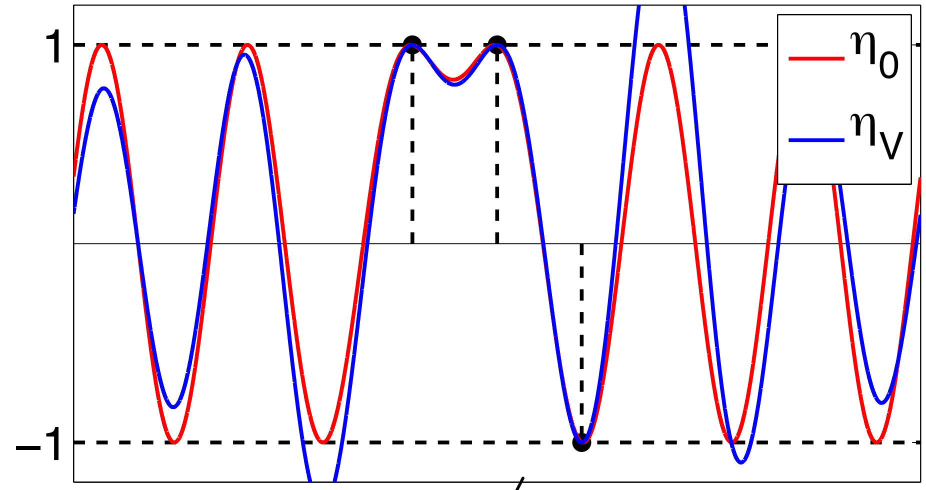
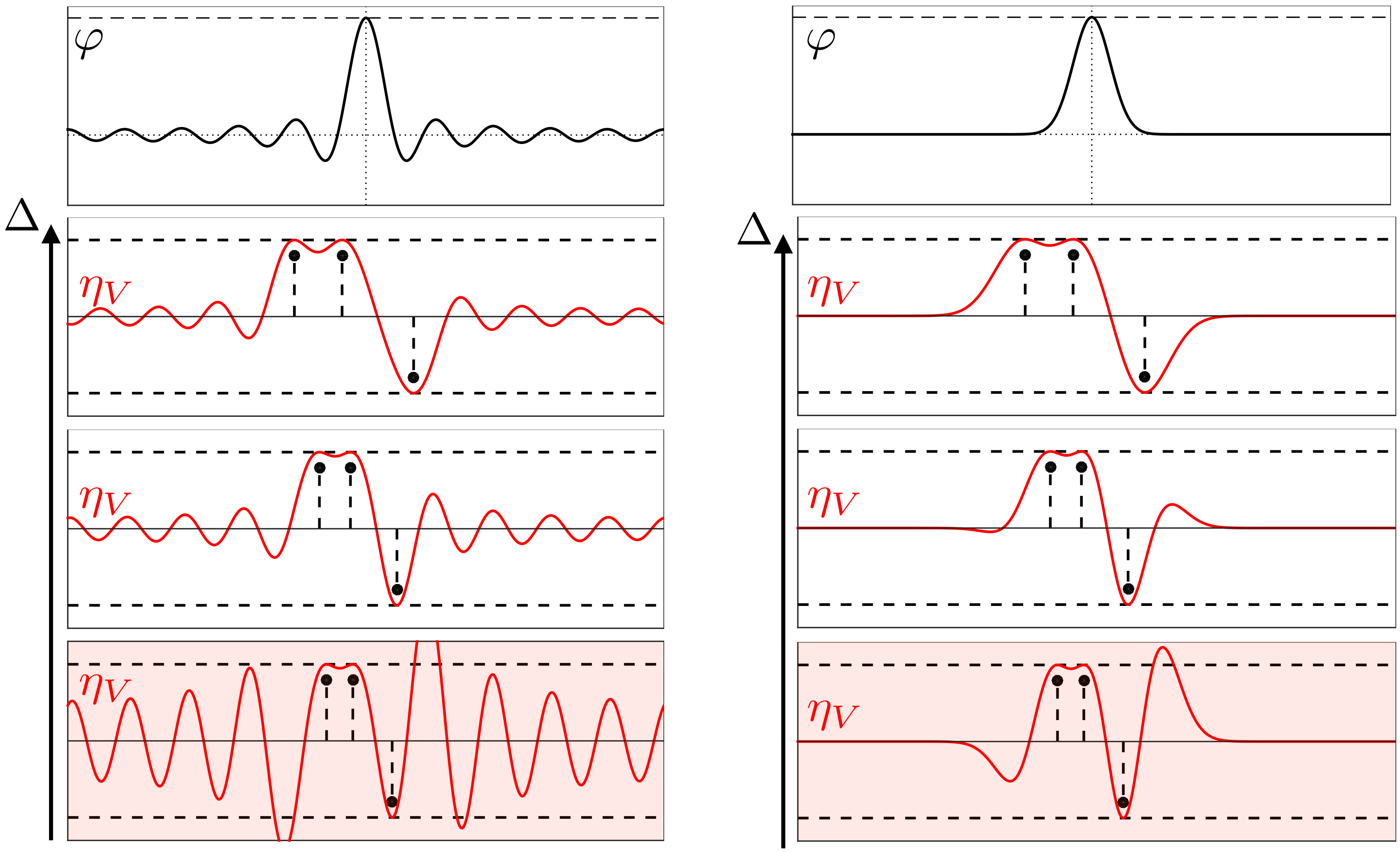


$\eta_0 \neq \eta_V$

When is η_V Non-degenerate ?

Input measure: $m_0 = m_{a_0,\Delta x_0}$, $\Delta \to 0$



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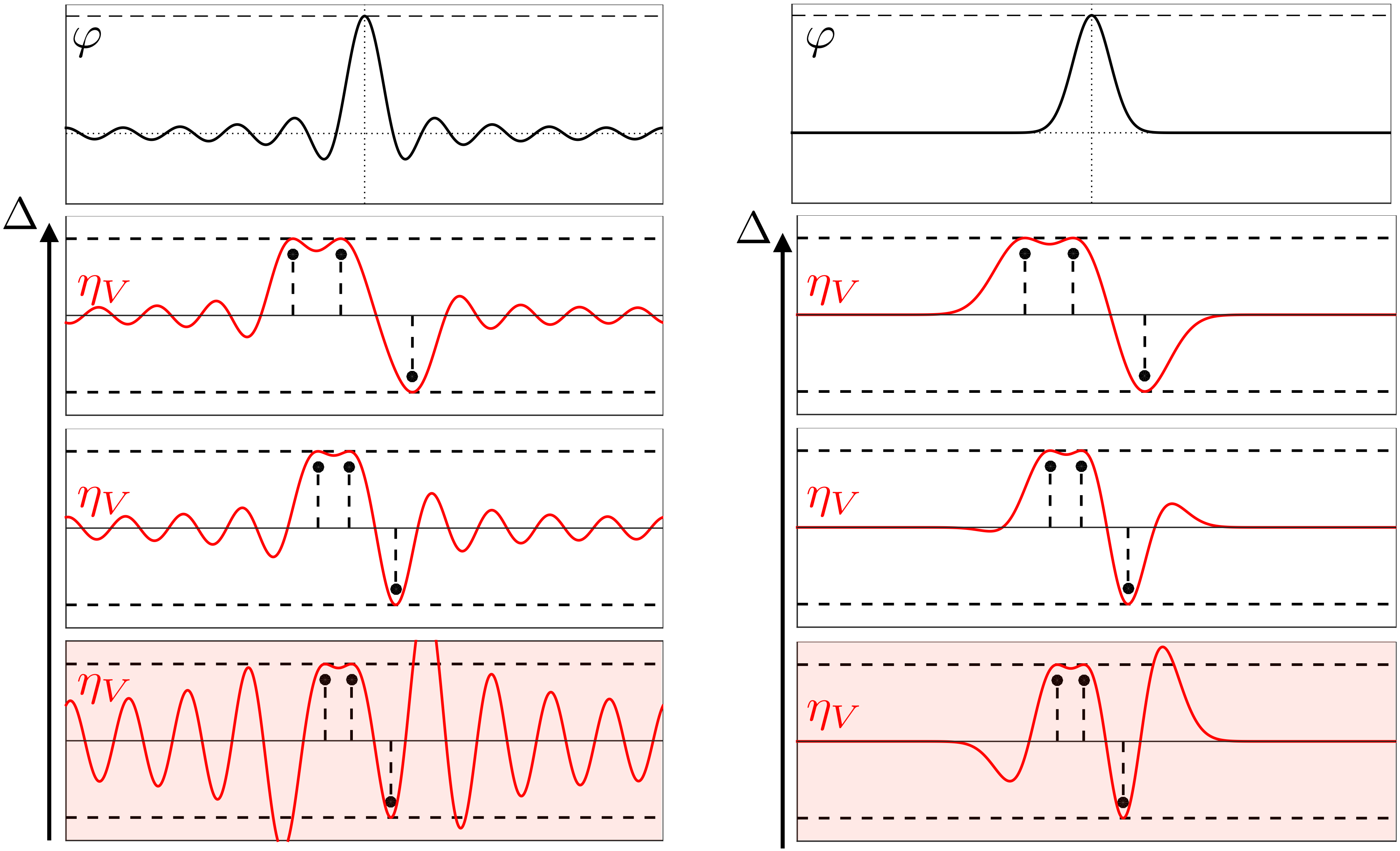


Theorem: [Tang, Recht, 2013]

$\exists C, (\Delta > C\sigma) \implies (\eta_V \text{ is non degenerate})$

Valid for:

- $\varphi(x) = e^{-x^2/\sigma^2}$
- $\varphi(x) = (1 + (x/\sigma)^2)^{-1}$
- $\dots$

Overview

- Sparse Spikes Super-resolution
- Robust Support Recovery
- **Asymptotic Positive Measure Recovery**
- Discrete vs. Continuous

Recovery of Positive Measures

$$m_0 = m_{a_0,x_0} \quad \text{where} \quad a_0 \in \mathbb{R}_+^N$$

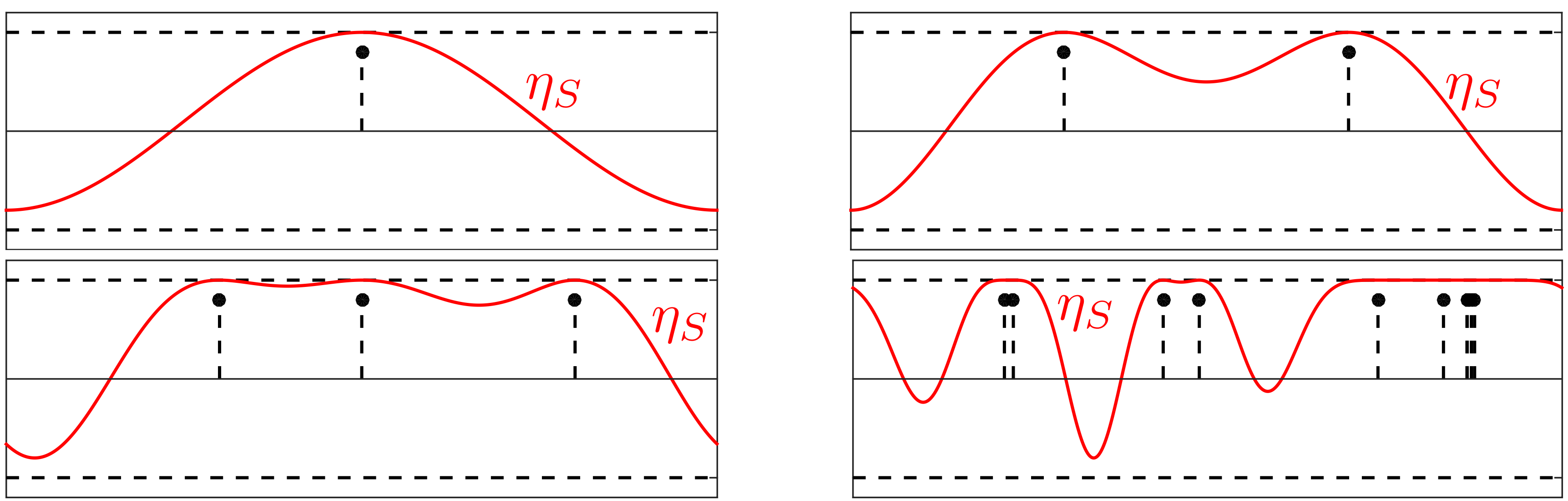
Theorem: let $\Phi m = \left(\int e^{-2i\pi k t} \mathrm{d}m(t)\right)_{k=-f_c}^{f_c}$ and

$$\eta_S(t) = 1 - \rho \prod_{i=1}^{N} \sin(\pi(t - x_i))^2$$

for $N \leqslant f_c$ and ρ small enough, $\eta_S \in \bar{\mathcal{D}}(m_0)$.

$\longrightarrow$ m_0 is recovered when there is no noise.

[de Castro et al. 2011]



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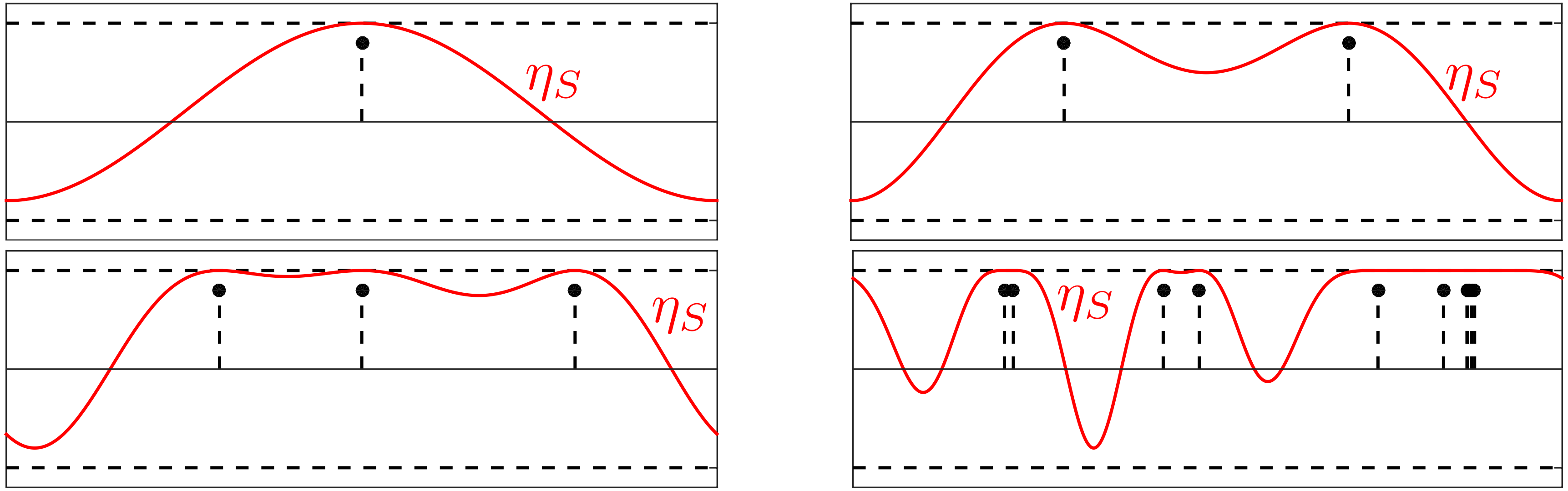

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[Morgenshtern, Candès, 2015] discrete ℓ^1 robustness.

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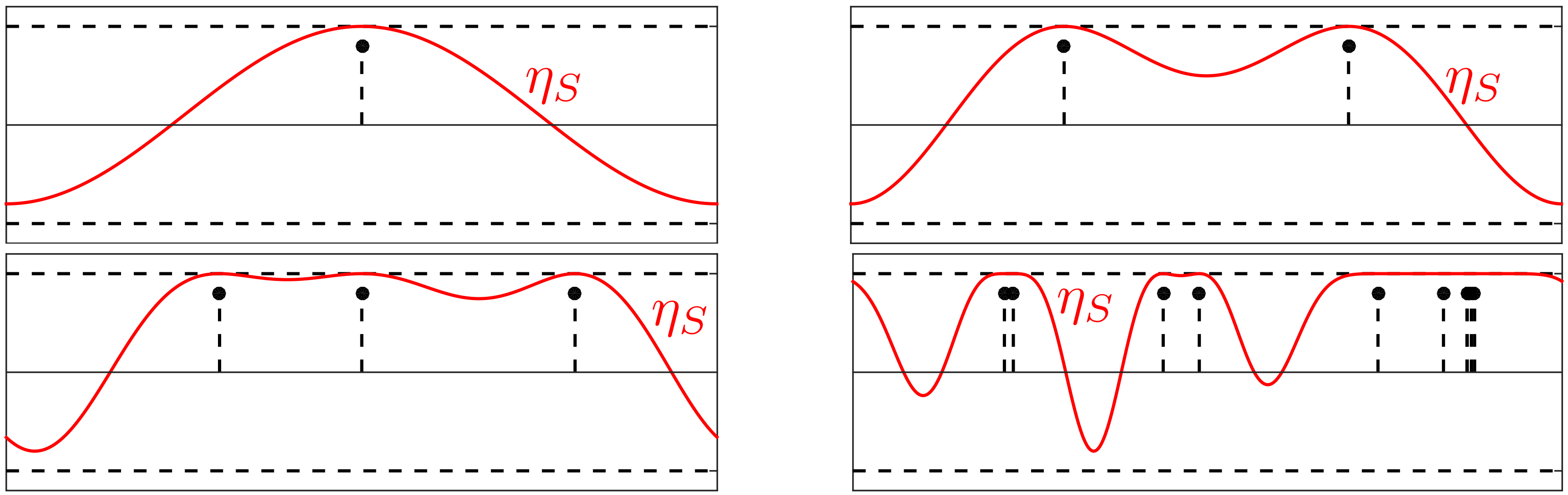

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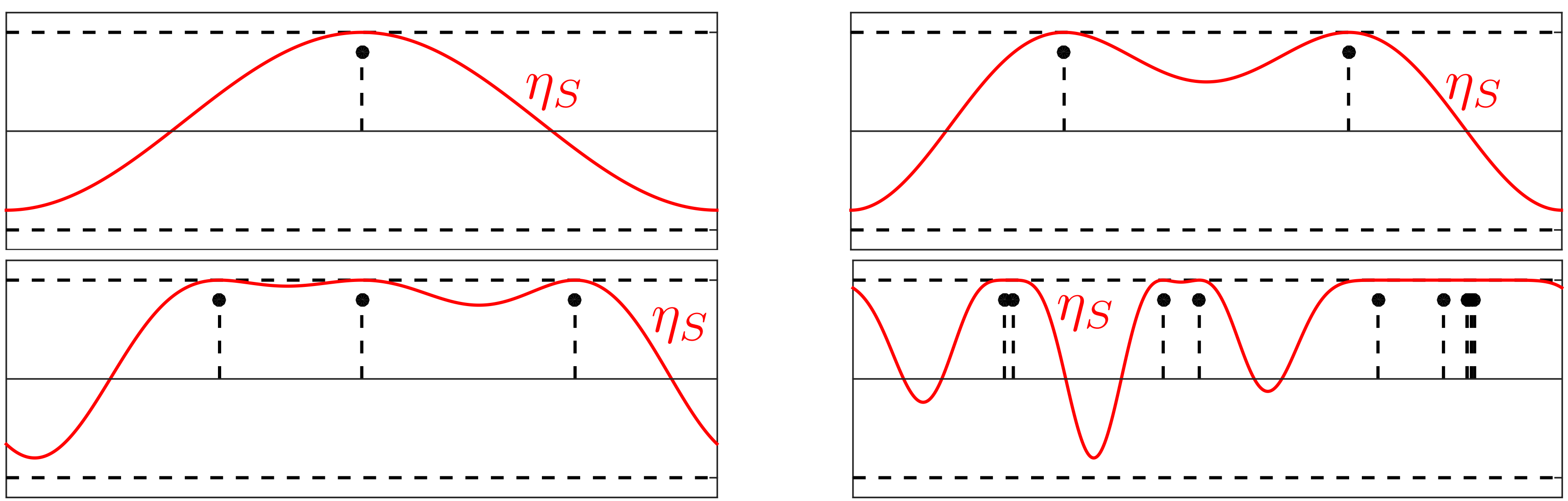
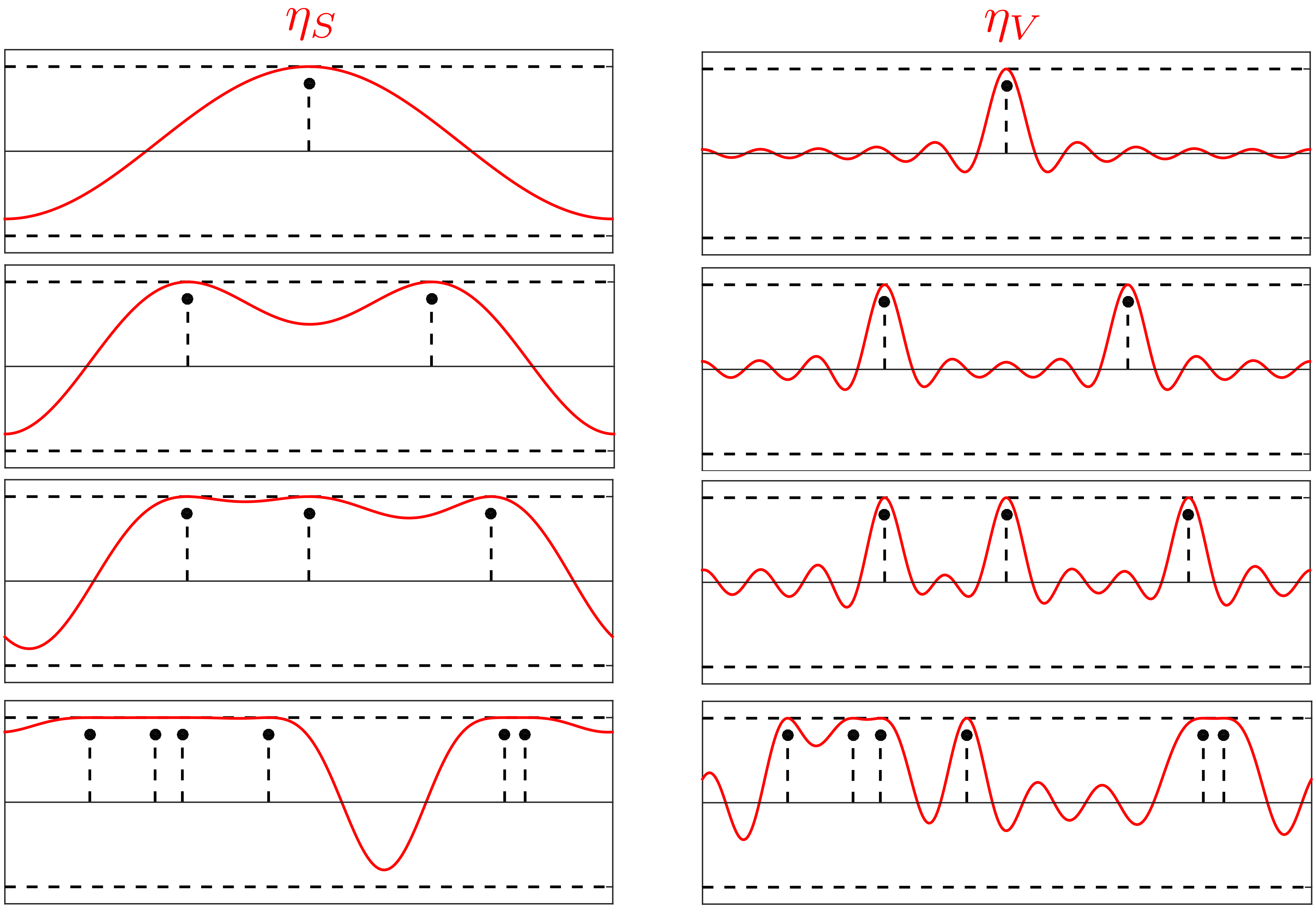
$\longrightarrow$ noise robustness of support recovery ?

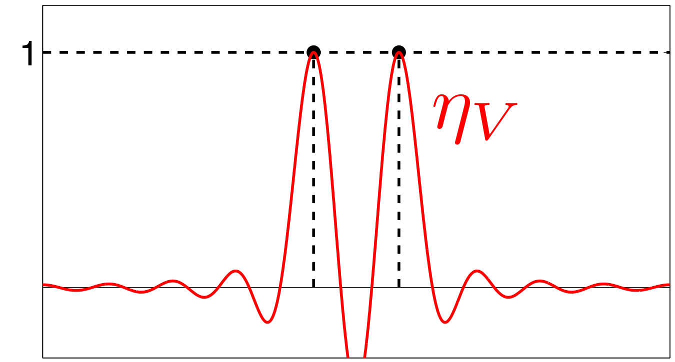


Comparison of Certificates



Asymptotic of Vanishing Certificate

$$m_0 = m_{a_0, \Delta x_0} \quad \text{where} \quad \Delta \to 0$$



Vanishing Derivative pre-certificate:

$$\eta_V \stackrel{\text{def.}}{=} \underset{\eta = \Phi^* p}{\operatorname{argmin}} \|p\| \quad \text{s.t.} \quad \forall\, i, \begin{cases} \eta(\Delta x_{0,i}) = 1, \\ \eta'(\Delta x_{0,i}) = 0. \end{cases}$$

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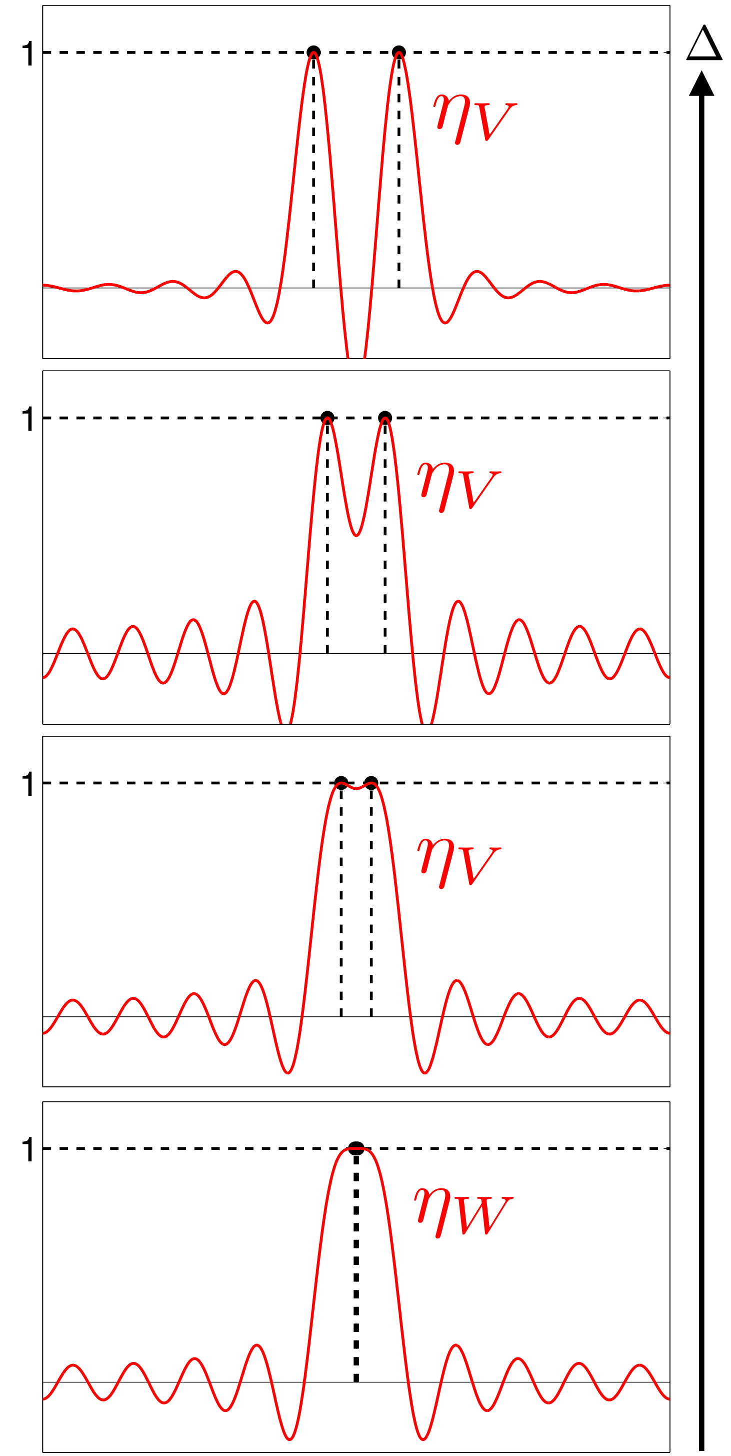

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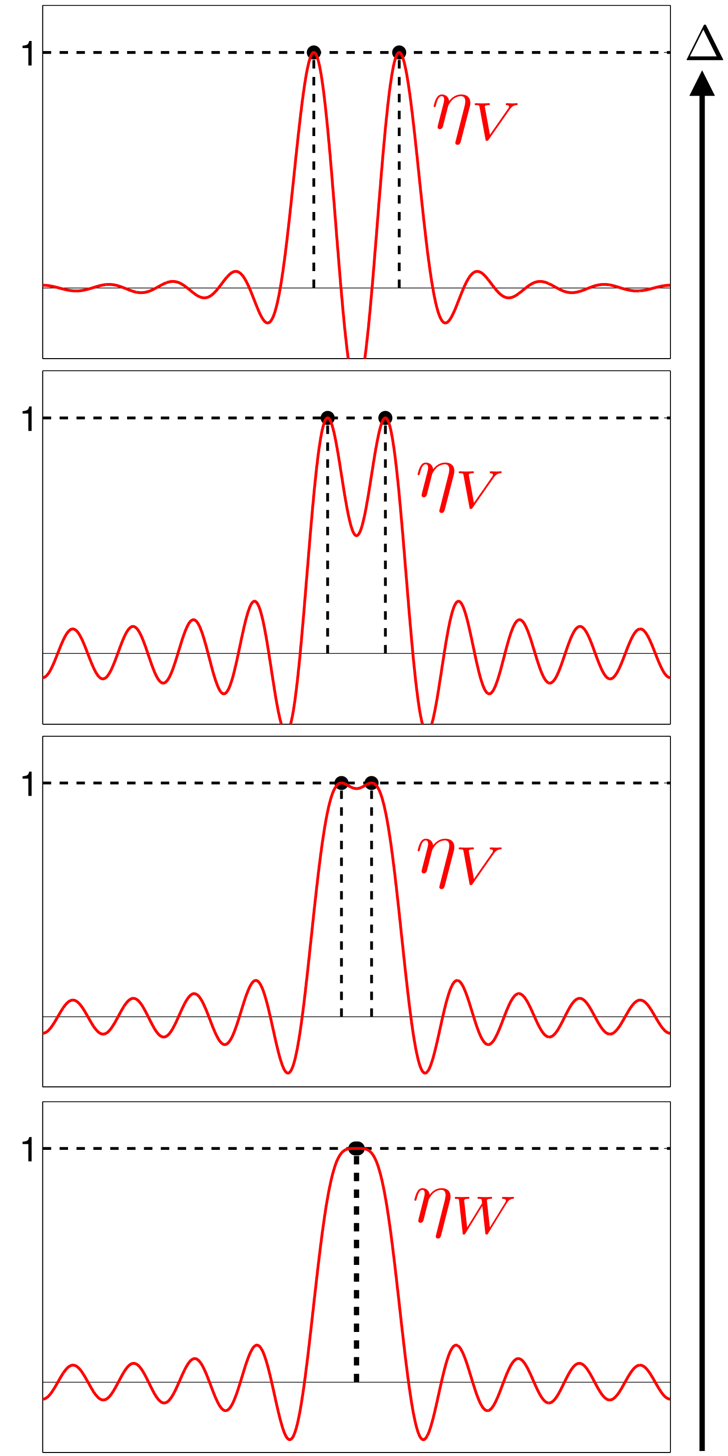
$\Delta \to 0$

Asymptotic pre-certificate:

$$\eta_W \overset{\text{def.}}{=} \underset{\eta=\Phi^* p}{\text{argmin}} \|p\|$$

$$\text{s.t.} \quad \begin{cases} \eta(0) = 1, \\ \eta'(0) = \ldots = \eta^{(2N-1)}(0) = 0. \end{cases}$$

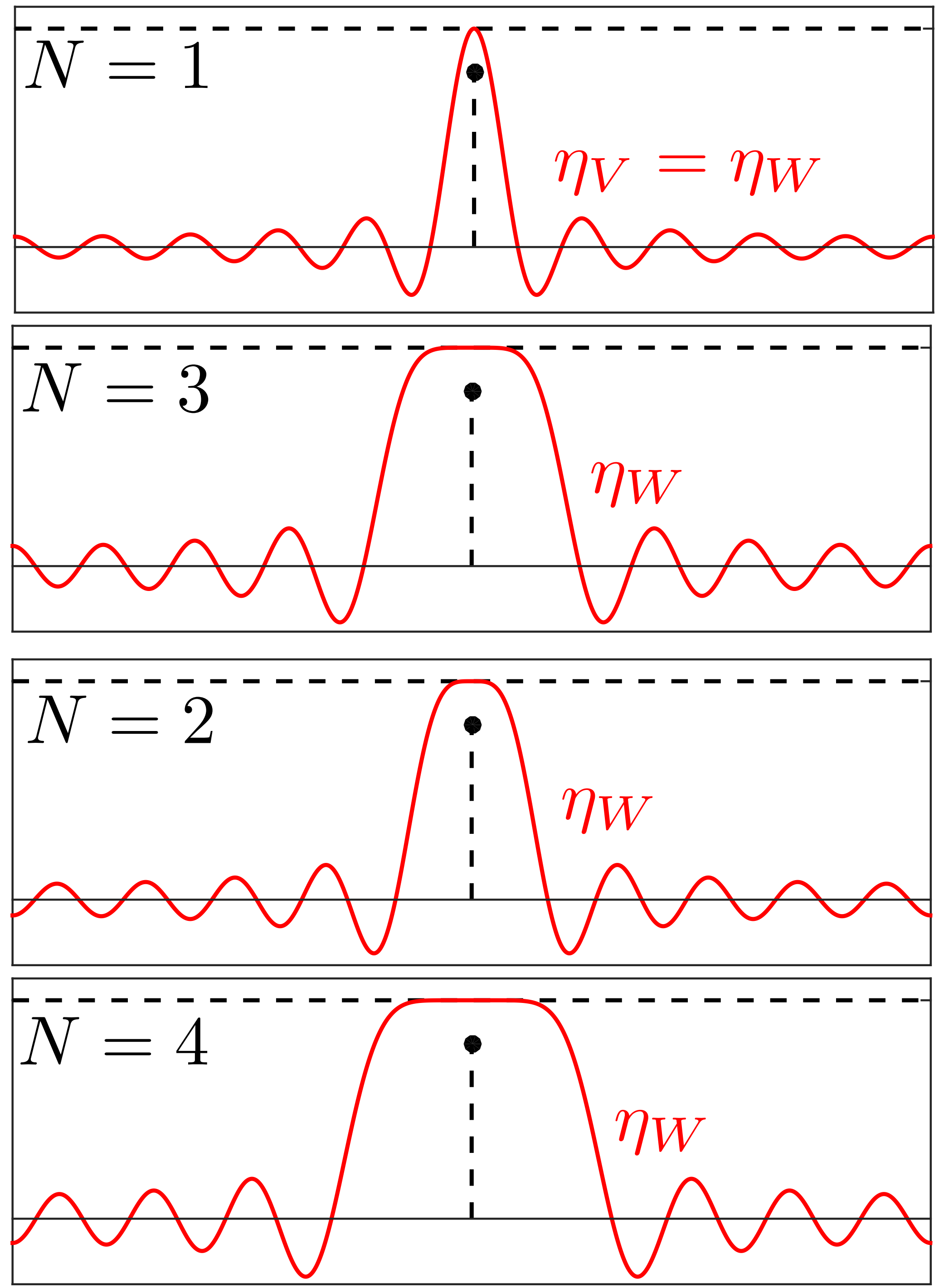


Asymptotic Certificate

$(2N-1)$-*Non degenerate:*

$\eta_W \in \bar{\mathcal{D}}_N$

$$\iff \begin{cases} \forall x \neq 0, |\eta_W(x)| < 1 \\ \eta_W^{(2N)}(0) \neq 0 \end{cases}$$



Asymptotic Certificate

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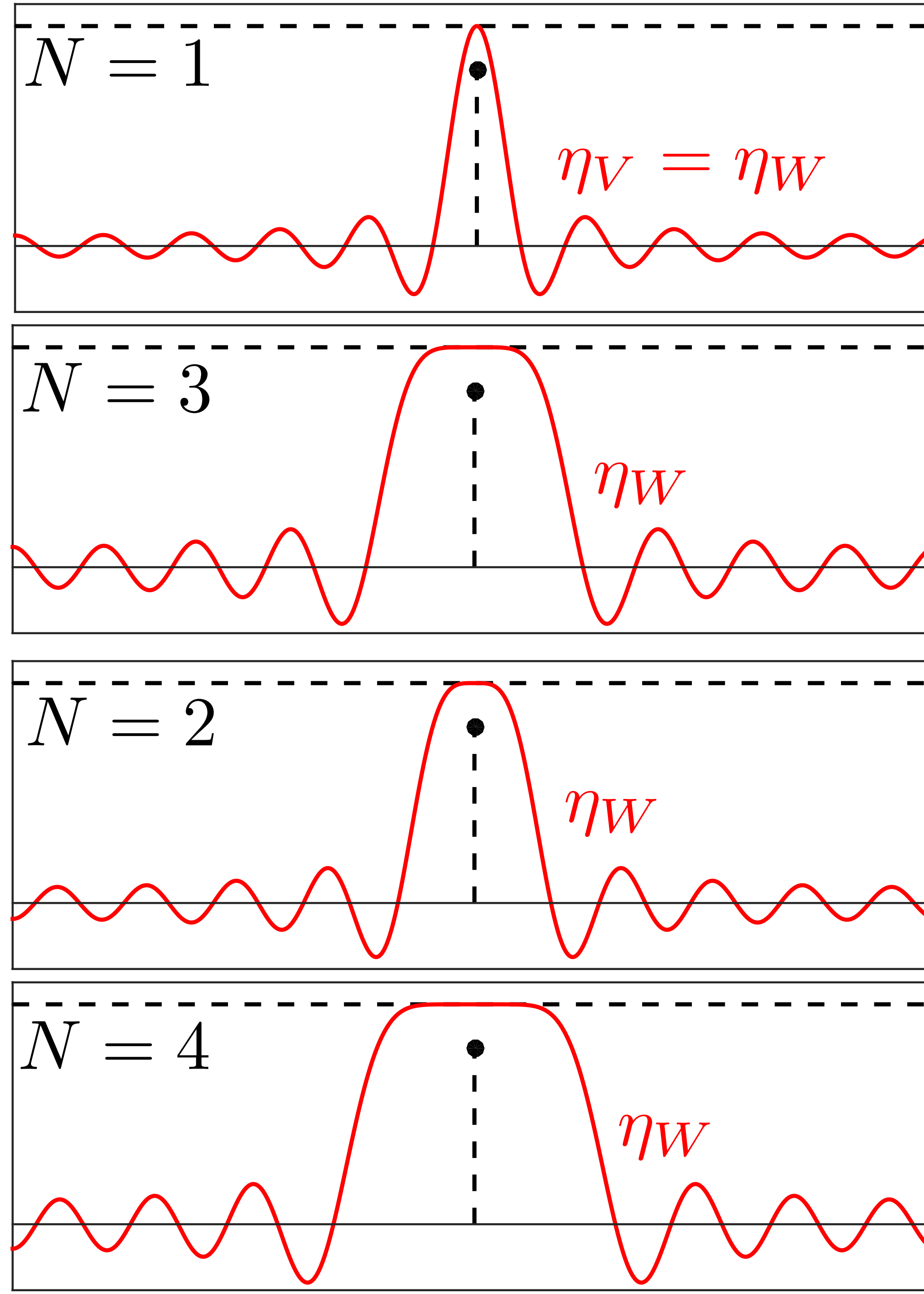
Lemma:

If $\eta_W \in \bar{\mathcal{D}}_N$, $\exists \Delta_0 > 0$,

$\forall \Delta < \Delta_0,\ \eta_V \in \bar{\mathcal{D}}(m_{\Delta x_0, a_0})$

$\rightarrow \eta_W$ govern stability as $\Delta \rightarrow 0$.



Asymptotic Robustness

[Denoyelle, D., P. 2014]

Theorem: If $\eta_W \in \bar{\mathcal{D}}_N$, letting $m_0 = m_{a_0, \Delta x_0}$, then

for $\left(\frac{w}{\lambda}, \frac{w}{\Delta^{2N-1}}, \frac{\lambda}{\Delta^{2N-1}}\right) = O(1)$

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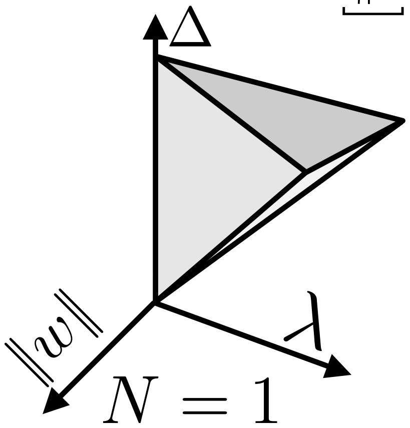

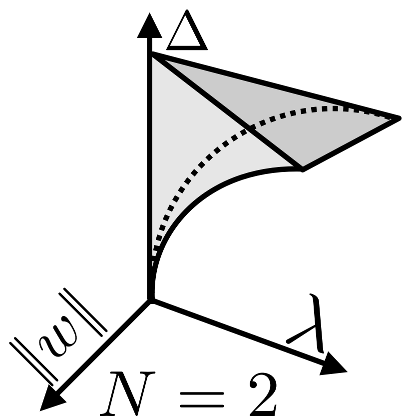

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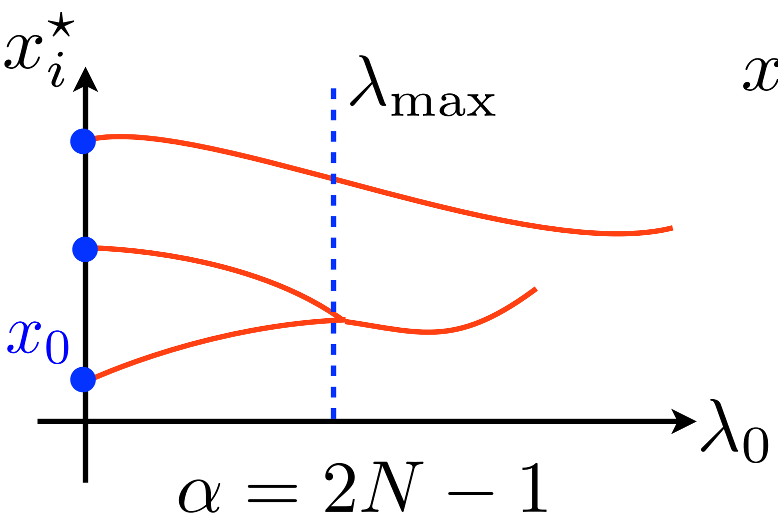
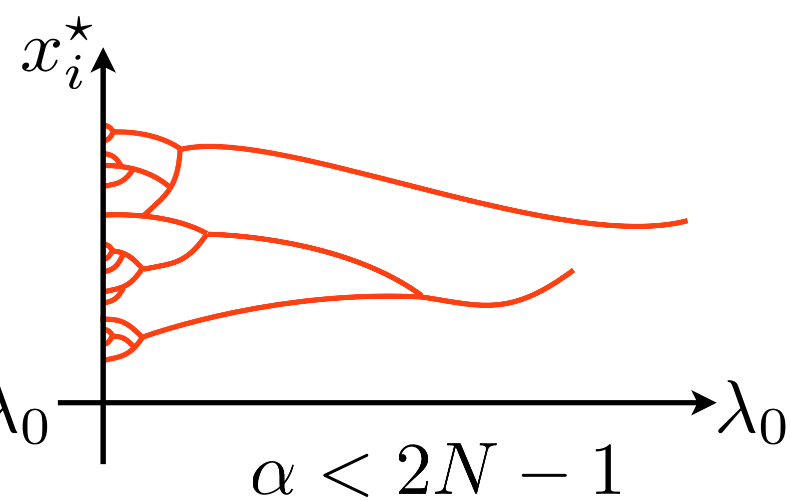
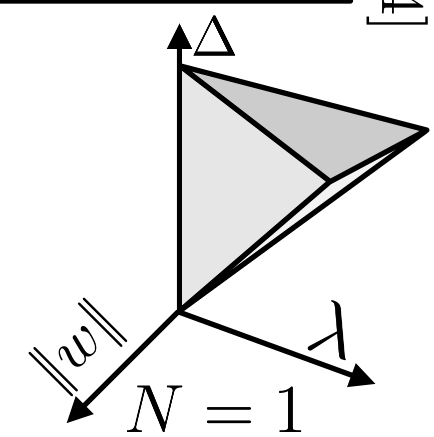
[Denoyelle, D., P. 2014]

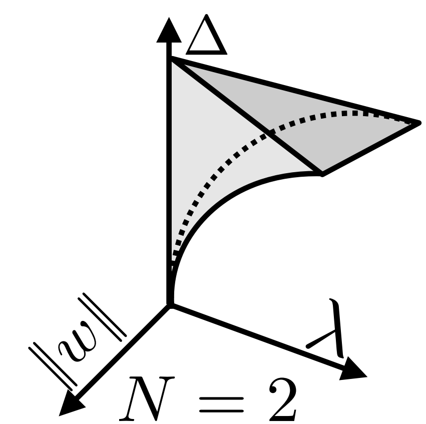
$$y = \Phi m_{a_0, t x_0} + w$$

Noise: $w = \lambda w_0$.
Regularization: $\lambda = \lambda_0 \Delta^\alpha$









When is η_W Non-degenerate ?

Proposition: one has $\eta_W^{(2N)}(0) < 0$. $\longrightarrow$ "locally" non-degenerate.

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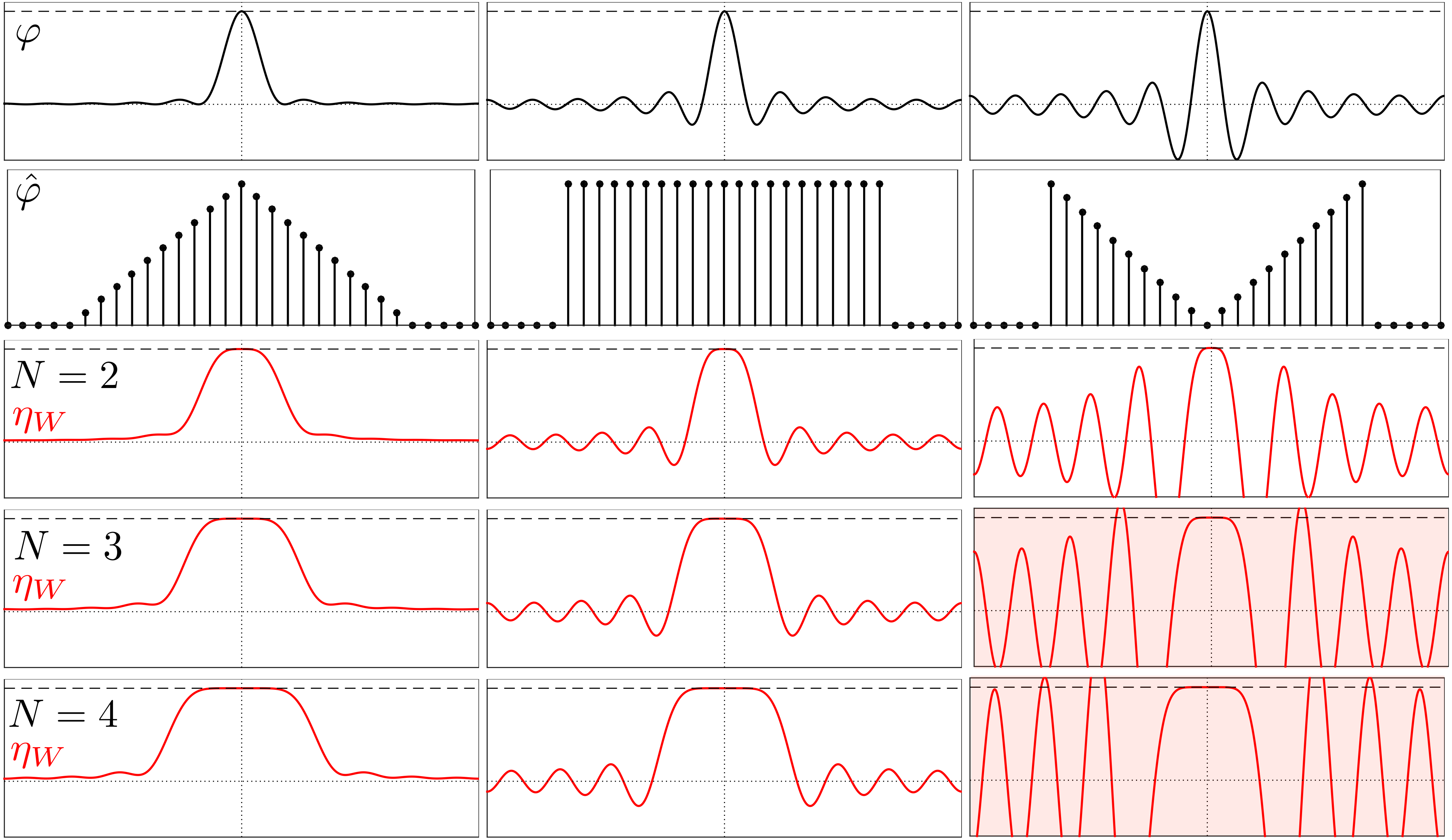

Overview

- Sparse Spikes Super-resolution
- Robust Support Recovery
- Asymptotic Positive Measure Recovery
- Discrete vs. Continuous

Discrete vs. Continuous Recovery

Measures: $$\min_{m} \frac{1}{2}\|\Phi(m) - y\|^2 + \lambda |m|(\mathbb{T})$$

On a grid z: $$\min_{a \in \mathbb{R}^N} \frac{1}{2}\|\Phi(m_{a,z}) - y\|^2 + \lambda \|a\|_1$$

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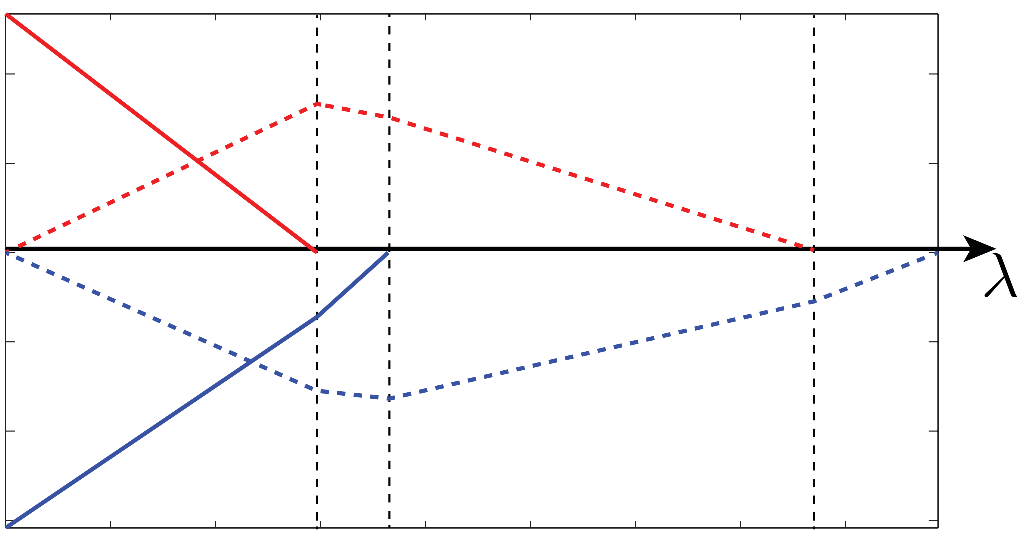
If $m_0 = m_{a_0,z}$, then for fine grids z, $\text{supp}(a^\star) \neq \text{supp}(a_0)$.

$\longrightarrow$ *Discrete support is not stable!*



Solution path $\lambda \mapsto a^\star$

Discrete Minimal Norm Certificate

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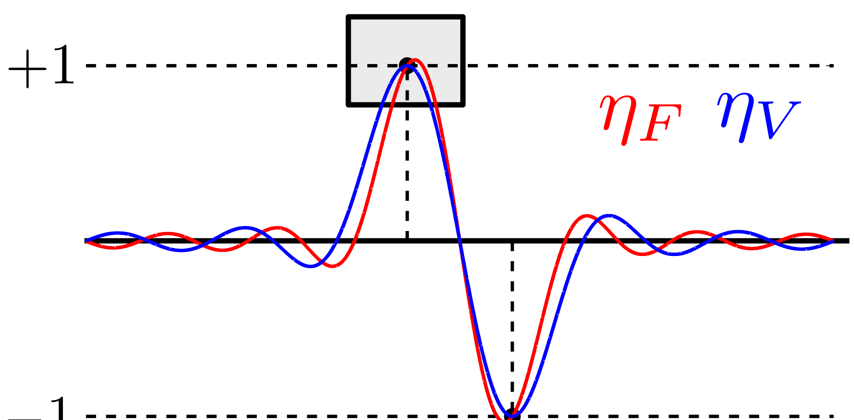

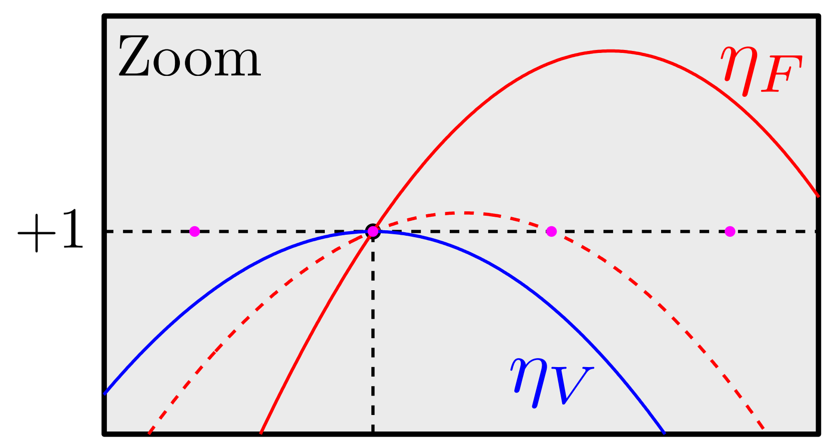

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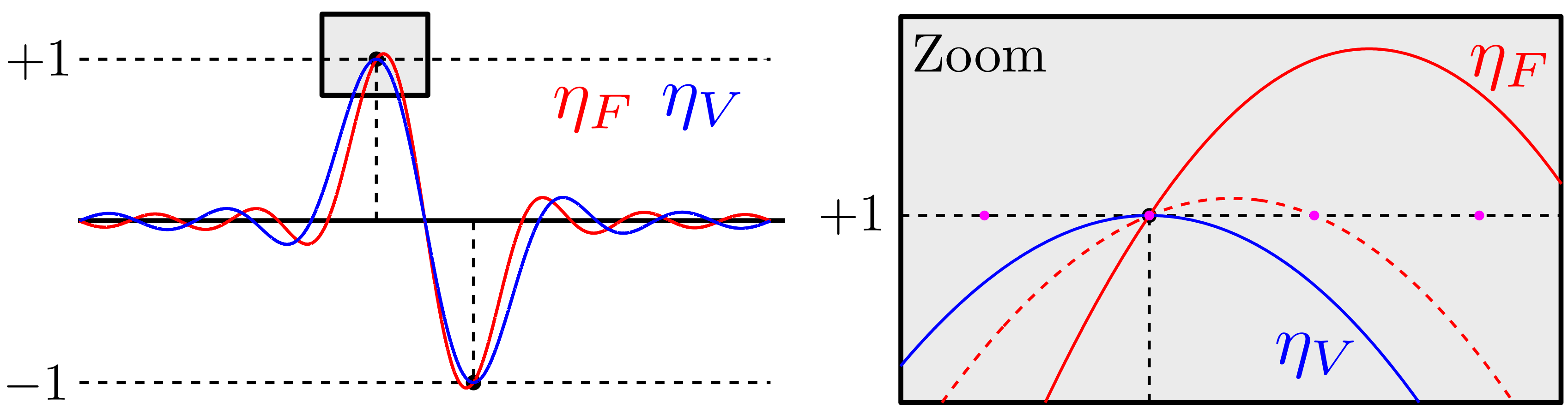

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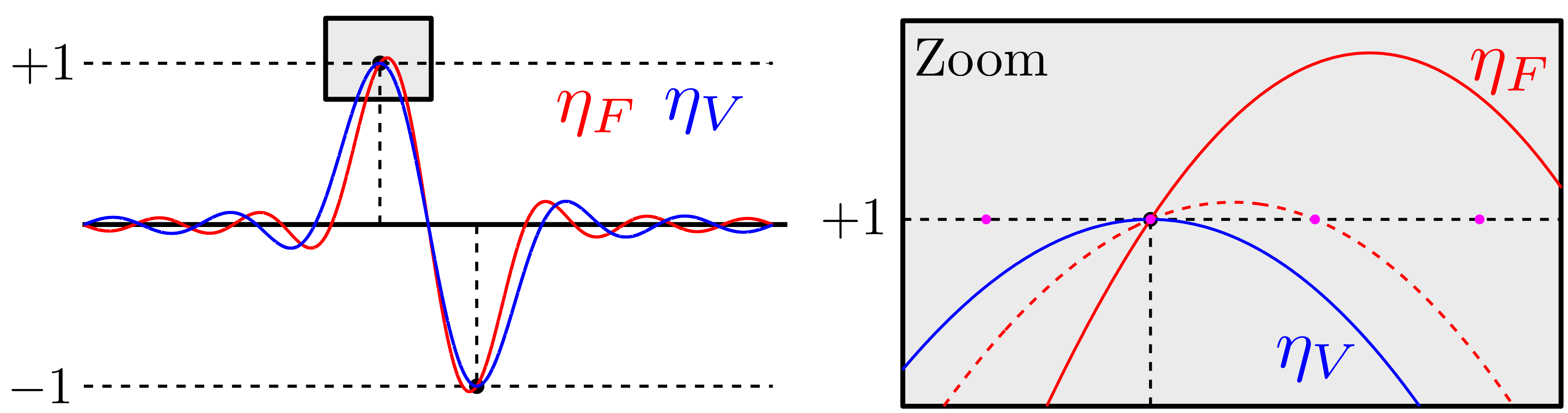

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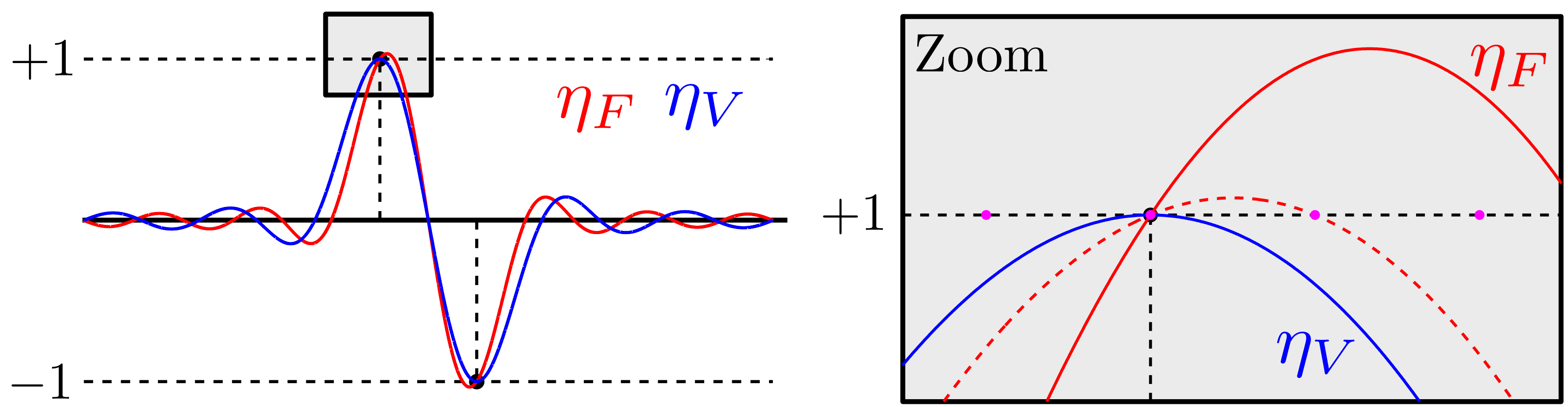


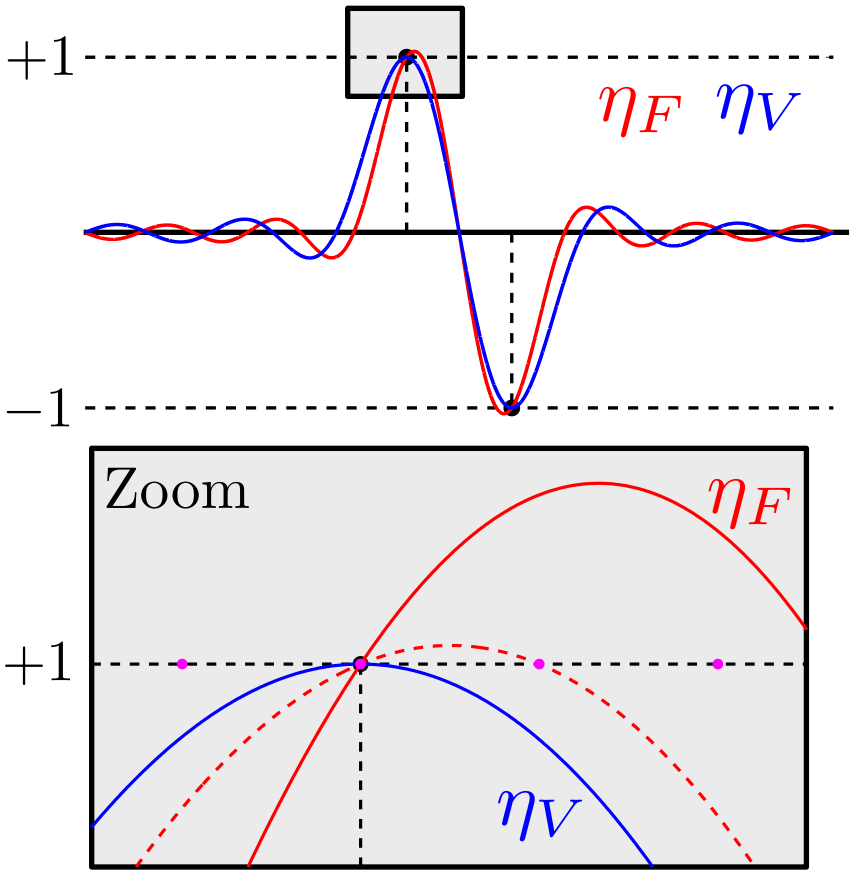
$\rightarrow \eta_F \notin \bar{\mathcal{D}}_d(m_0)$ for fine grids!

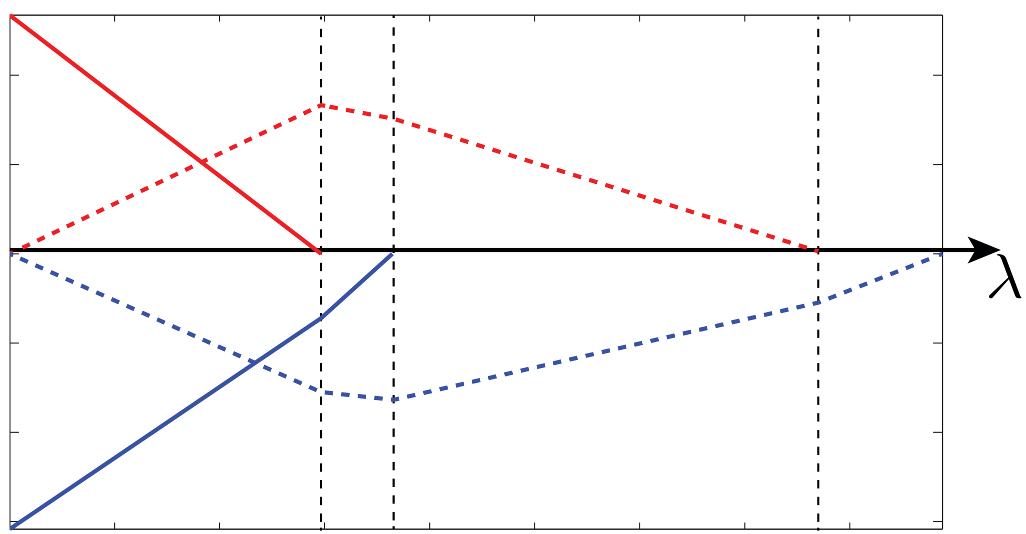
Structure of Discrete Recovered Support

Theorem: if $\eta_0 \in \bar{\mathcal{D}}(m_0)$,

then for fine grids and for $(\|w\|/\lambda, \lambda) = O(1)$,

$\text{supp}(a^\star) \subset \text{supp}(a_0) \oplus \{-1, 0, 1\}$





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Conclusion

Deconvolution of measures:

$\longrightarrow L^2$ errors are not well-suited.

- Weak-* convergence.
- Optimal transport distance.
- Exact support estimation.
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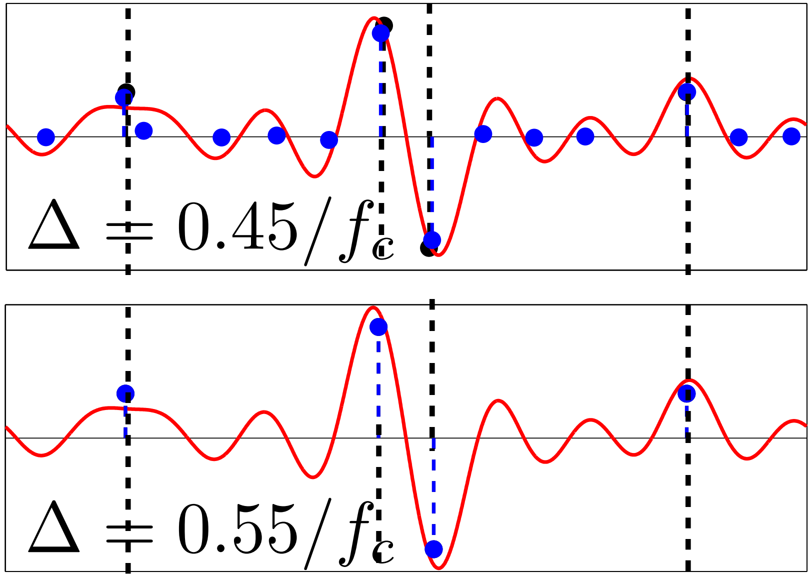

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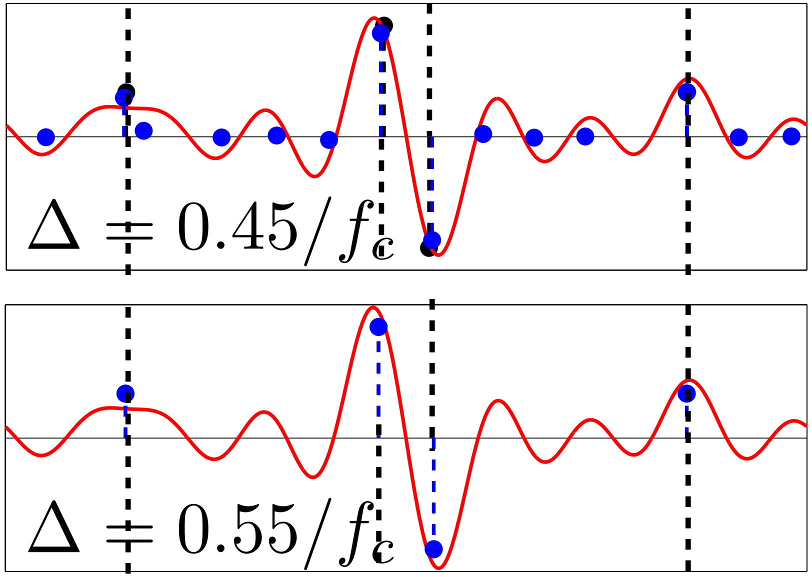

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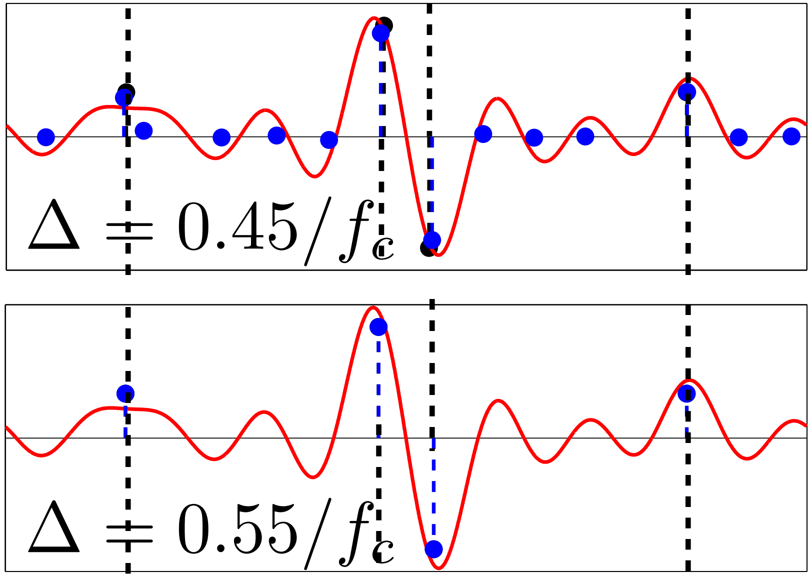

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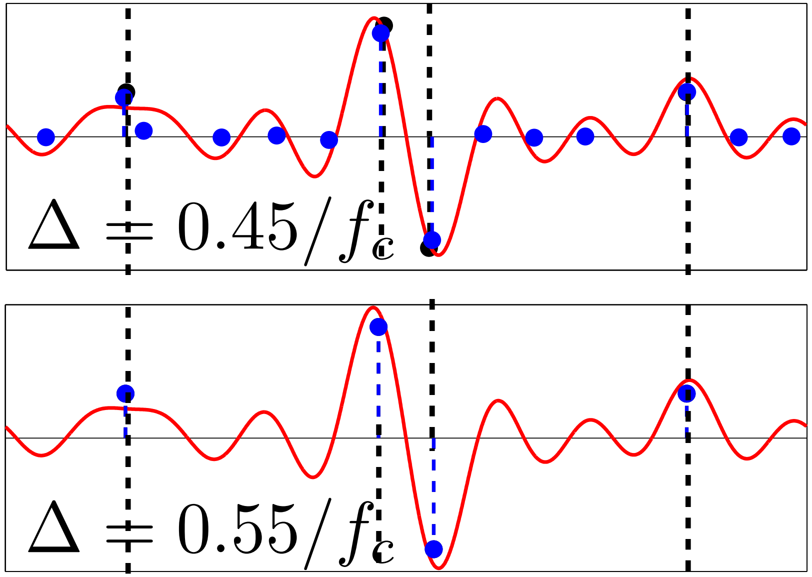

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Open problem: other regularizations (e.g. piecewise constant) ?