Extended Proof of Steps 2(b) and 4(b)

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This document serves as an extended proof $\mathcal{H}_1$ and $\mathcal{H}_{t+1}$ of Lemma 5, part (b).(i) from *Capacity-achieving Sparse Superposition Codes via Approximate Message Passing Decoding* [1]. References to [1] and the proof of Lemma 5 within, will be made throughout this document.

Lemma 5 (b)(i) *We will show that the following statement holds for* $0 \leq t \leq T^*$, *where* $T^* = \left\lceil \frac{2\mathcal{C}}{\log(\mathcal{C}/R)} \right\rceil$ *assumed to be less than* n. *Consider the following functions defined on* $\mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}$. *For* $x, y, z \in \mathbb{R}^M$ *and* $\ell \in [L]$, *let*

$$\begin{aligned}
&\phi_{1,\ell}(x,y,z) := x^* y/M, && \\
&\phi_{2,\ell}(x,y,z) := \|\eta^r_\ell(z-x)\|^2/\log M, && 0 \leq r \leq t, \\
&\phi_{3,\ell}(x,y,z) := [\eta^r_\ell(z-x) - z]^*[\eta^s_\ell(z-y) - z]/\log M, && -1 \leq r \leq s \leq t, \\
&\phi_{4,\ell}(x,y,z) := y^*[\,\eta^r_\ell(z-x) - z]/\log M, && -1 \leq r \leq t,
\end{aligned} \tag{1}$$

where for $r \geq 0$, $\eta^r_\ell(\cdot)$ *is the restriction of* η^r *to section* ℓ, *i.e., for* $x \in \mathbb{R}^M$,

$$\eta^r_{\ell,i}(x) := \sqrt{nP_\ell}\, \frac{\exp\left(\frac{x_i \sqrt{nP_\ell}}{\tau_r^2}\right)}{\sum_{j=1}^M \exp\left(\frac{x_j \sqrt{nP_\ell}}{\tau_r^2}\right)}, \;\; i = 1, \dots, M.$$

(Also, $\eta^{-1}_{\ell,i}(\cdot) := 0$ *for* $i \in [M]$.*) Then, for* $k \in \{1,2,3,4\}$ *and arbitrary constants* $(a_0, \dots, a_t, b_0, \dots, b_t)$, *we have*

$$\lim n^\delta \left| \frac{1}{L} \sum_{\ell=1}^L \phi_{k,\ell}\left(\sum_{r=0}^t a_r h^{r+1}_\ell, \sum_{s=0}^t b_s h^{s+1}_\ell, \beta_{0_\ell} \right) - c_k \right| \overset{a.s.}{=} 0, \tag{2}$$

where

$$c_k := \lim \frac{1}{L} \sum_{\ell=1}^L \mathbb{E}\left[\phi_{k,\ell}\left(\sum_{r=0}^t a_r \bar{\tau}_r \breve{Z}_{r_\ell}, \sum_{s=0}^t b_s \bar{\tau}_s \breve{Z}_{s_\ell}, \beta_\ell \right)\right].$$

Here $\breve{Z}_0, ..., \breve{Z}_t$ *are length-*N *Gaussian random vectors independent of* β, *with* $\breve{Z}_{r_\ell}$ *denoting the* ℓ*th section of* $\breve{Z}_r$. *For* $0 \leq s \leq t$, $\{\breve{Z}_{s_j}\}_{j \in [N]}$ *are i.i.d.* $\sim \mathcal{N}(0,1)$, *and for each* $i \in [N]$, $(\breve{Z}_{0_i}, \dots, \breve{Z}_{t_i})$ *are jointly Gaussian with* $\mathbb{E}[\bar{\tau}_r \breve{Z}_{r_i} \bar{\tau}_t \breve{Z}_{t_i}] = \bar{\tau}_t^2$ *for* $0 \leq r \leq t$. *Both limits in* (2) *exists and are finite for each* $\phi_{k,\ell}$ *in* (1).

In the proof, we will use Lemmas A.1 and A.2 which are stated and proved in the Appendix.

1 Step 2: Showing $\mathcal{H}_1$(b).(i) holds

We will show that result (2) holds when $t = 0$.

Proof. Consider the functions $\phi_{k,\ell}(x,y,z)$ for $k \in \{1,2,3,4\}$ defined in (1). First note that the result of this Lemma, given in (2), is true for an additional group of functions defined as follows:

$$\phi_{5,\ell}(x,y,z) = \mathbb{E}_Z\left\{\phi_{k,\ell}(x+\sigma_Z Z, y+\sigma_Z Z, z)\right\},$$

where $k = \{1,2,3,4\}$, random vector $Z \in \mathbb{R}^M$ is i.i.d. $\sim \mathcal{N}(0,1)$ independent of x,y,z, and $\lim \sigma_Z \overset{a.s.}{=} constant$. We do not prove the result explicitly for these functions since it follows by application of Jensen's Inequality. In what follows we prove the result for generic $\phi_{k,\ell}(x,y,z)$ with $k \in \{1,2,3,4\}$ and we state explicitly which k we refer to when it is important to the results.

From [1, Lemma 4] it follows,

$$\phi_{k,\ell}(a_0 h^1_\ell,\, b_0 h^1_\ell,\, \beta_{0_\ell})|_{\mathscr{S}_{1,0}} \overset{d}{=} \phi_{k,\ell}\left(a_0\bar{\tau}_0 Z_{0_\ell} + a_0[\Delta_{1,0}]_\ell,\, b_0\bar{\tau}_0 Z_{0_\ell} + b_0[\Delta_{1,0}]_\ell,\, \beta_{0_\ell}\right).$$

For the first step, we show that the deviation term $\Delta_{1,0}$ can be dropped when considering the limit. Define

$$\mathsf{diff}_{1,k,\ell} := \phi_{k,\ell}\left(a_0\bar{\tau}_0 Z_{0_\ell} + a_0[\Delta_{1,0}]_\ell,\, b_0\bar{\tau}_0 Z_{0_\ell} + b_0[\Delta_{1,0}]_\ell,\, \beta_{0_\ell}\right) - \phi_{k,\ell}\left(a_0\bar{\tau}_0 Z_{0_\ell},\, b_0\bar{\tau}_0 Z_{0_\ell},\, \beta_{0_\ell}\right). \quad (3)$$

Considering t fixed, for each function in (1) we first show the following.

$$\lim \frac{n^\delta}{L}\sum_{\ell=1}^{L} |\mathsf{diff}_{1,k,\ell}| \overset{a.s.}{=} 0. \quad (4)$$

To prove the above we will supply upper bounds on the difference $\mathsf{diff}_{1,k,\ell}$ defined in (3) which approach 0 almost surely in the limit. To do so, we consider each k separately. The following two results are useful:

1. From [1, $\mathcal{H}_1$(a)], for each $\ell \in [L]$,

$$\max_{j\in sec(\ell)} |[\Delta_{1,0}]_j| \overset{a.s.}{=} \Theta(n^{-\delta'}\sqrt{\log M}), \quad \text{and} \quad \max_{j\in sec(\ell)} |h^1_j| \leq c_1\sqrt{\log M}, \quad (5)$$

   where $c_1 > 0$ is a constant not depending on N.

2. The following is due to [1, Fact 7]. For $Z_1, Z_2, \ldots$ i.i.d. $\sim \mathcal{N}(0,1)$, with probability 1 we have

$$\max_{j\in[M]} |Z_j| \leq \sqrt{2K\log M} \text{ for all sufficiently large } M. \quad (6)$$

k = 1. Because of the $\Delta_{1,0}$ bounds given in (5), it follows from Lemma (A.1) for each $\ell \in [L]$,

$$|\mathsf{diff}_{1,k,\ell}| \overset{a.s.}{\leq} \frac{C\sqrt{\log M}}{n^{\delta'}}\left[|a_0\bar{\tau}_0| \max_{j\in sec(\ell)}|Z_{0_j}| + |b_0\bar{\tau}_0| \max_{j'\in sec(\ell)}|Z_{0_j}| + \frac{\sqrt{\log M}}{n^{\delta'}}\right],$$

for some $\delta' > 0$. Considering (6) and the above, $\frac{n^\delta}{L}\sum_{\ell=1}^{L}|\mathsf{diff}_{1,k,\ell}| \overset{a.s.}{\leq} C' n^{\delta-\delta'}\log M$, which approaches 0 when $\delta < \delta'$.

k = 2, 3. Because of the $\Delta_{1,0}$ bounds given in (5), it follows from Lemma (A.2) for each $\ell \in [L]$, $|\mathsf{diff}_{1,k,\ell}| \overset{a.s.}{\leq} C\log M n^{-\delta'}$, for some $\delta' > 0$. Now plugging this into (4) we establish the following upper bound:

$$\frac{n^\delta}{L}\sum_{\ell=1}^{L}|\mathsf{diff}_{1,k,\ell}| \overset{a.s.}{\leq} \frac{n^\delta}{L}\sum_{\ell=1}^{L}\frac{C\log M}{n^{\delta'}} \leq Cn^{\delta-\delta'}\log M.$$

The right-side of the above approaches 0 when $\delta < \delta'$.

k = 4. We first establish an upper bound for $\phi_{4,\ell}(x, y, z)$.

$$
\begin{aligned}
(\log M)|\phi_{k,\ell}\,(x, y, z) &- \phi_{4,\ell}(x + \Delta_x, y + \Delta_y, z)| \\
&= |y^*[\eta^r_\ell(z - x) - z] - (y + \Delta_y)^*[\eta^r_\ell(z - x - \Delta_x) - z]| \\
&\leq |y^*[\eta^r_\ell(z - x) - \eta^r_\ell(z - x - \Delta_x)]| + |\Delta^*_y[\eta^r_\ell(z - x - \Delta_x) - z]| \\
&\leq \frac{c(\log M)^{3/2}}{n^{\delta'}} \max_{k\in[M]}|y_k| + \max_{j\in[M]}|\Delta_{y_j}| \sum_{i=1}^{M} |\eta^r_{\ell_i}(z - x - \Delta_x) - z_i|.
\end{aligned} \tag{7}
$$

The last line in the above follows from Lemma A.2, for some constant $c > 0$. Now to prove (4) we use the above bound applied to $\mathsf{diff}_{1,k,\ell}$ defined in (3).

$$
|\mathsf{diff}_{1,k,\ell}| \overset{a.s.}{\leq} \frac{c(\log M)^{1/2}}{n^{\delta'}}|b_0\bar{\tau}_0| \max_{j'\in sec(\ell)}|Z_{0_{j'}}| + \frac{2\sqrt{nP_\ell}}{\log M} \max_{j\in sec(\ell)}|[\Delta_{1,0}]_j|,
$$

where we have used the fact that $\sum_{i=1}^{M}|\eta^r_{\ell_i}(\beta_{0_\ell} - a_0\bar{\tau}_0 Z_{0_\ell} - a_0[\Delta_{1,0}]_\ell) - \beta_{0_i}| \leq 2\sqrt{nP_\ell}$ for each $\ell \in [L]$ and $0 \leq r \leq t$. Now using the above and the results stated in (5) and (6) we find, $\frac{n^\delta}{L}\sum_{\ell=1}^{L}|\mathsf{diff}_{1,k,\ell}| \overset{a.s.}{\leq} c'n^{\delta-\delta'}\log M + c''n^{\delta-\delta'}$, which approaches 0 when $\delta < \delta'$.

In what follows we are justified in dropping the deviation terms $\Delta_{1,0}$. For the second step of the proof, we will appeal to the Strong Law for Triangular Arrays [1, Fact 2] which will show

$$
\lim n^\delta \left[\frac{1}{L}\sum_{\ell=1}^{L}\phi_{k,\ell}\,(a_0\bar{\tau}_0 Z_{0_\ell},\, b_0\bar{\tau}_0 Z_{0_\ell},\, \beta_{0_\ell}) - \frac{1}{L}\sum_{\ell=1}^{L}\mathbb{E}_{Z_0}\left\{\phi_{k,\ell}\,(a_0\bar{\tau}_0 Z_{0_\ell},\, b_0\bar{\tau}_0 Z_{0_\ell},\, \beta_{0_\ell})\right\}\right] \overset{a.s.}{=} 0. \tag{8}
$$

Let $\tilde{Z}_0$ be an independent copy of Z_0 and define for each $k = \{1, 2, 3, 4\}$,

$$
\mathsf{diff}_{2,k,\ell} := \phi_{k,\ell}\left(a_0\bar{\tau}_0\tilde{Z}_{0_\ell},\, b_0\bar{\tau}_0\tilde{Z}_{0_\ell},\, \beta_{0_\ell}\right) - \phi_{k,\ell}\,(a_0\bar{\tau}_0 Z_{0_\ell},\, b_0\bar{\tau}_0 Z_{0_\ell},\, \beta_{0_\ell})\,. \tag{9}
$$

In order to use the Strong Law for Triangular Arrays [1, Fact 2] to get result (8) we will prove the following for each function in (1).

$$
\frac{1}{L}\sum_{\ell=1}^{L}\mathbb{E}_{\tilde{Z}_0,Z_0}\left|n^\delta\mathsf{diff}_{2,k,\ell}\right|^{2+\kappa} \leq cL^{\kappa/2}, \tag{10}
$$

for some $0 \leq \kappa \leq 1$ and c some constant. Note that the exact requirement of [1, Fact 2] follows from (10) by an application of Jensen's Inequality. To prove (10), we consider each k separately. The following result, which follows from (6) for sufficiently large M, will be useful. Let $Z \in \mathbb{R}^M$ be a vector of i.i.d. standard Gaussian random variables,

$$
\left(\max_{j\in[M]}|Z_j|\right)^{2+\kappa} = \max_{j\in[M]}|Z_j|^{2+\kappa} \overset{a.s}{=} \Theta\left(\sqrt{\log M}^{2+\kappa}\right), \tag{11}
$$

k=1. First we upper bound the difference in (9) as follows:

$$
|\mathsf{diff}_{2,k,\ell}|^{2+\kappa} = |a_0 b_0\bar{\tau}_0^2|^{2+\kappa}\frac{\left|\|\tilde{Z}_{0_\ell}\|^2 - \|Z_{0_\ell}\|^2\right|^{2+\kappa}}{M^{2+\kappa}} \leq |a_0 b_0\bar{\tau}_0^2|^{2+\kappa}\left(\max_{j\in sec(\ell)}|\tilde{Z}_{0_j}|^{2(2+\kappa)} + \max_{j\in sec(\ell)}|Z_{0_j}|^{2(2+\kappa)}\right). \tag{12}
$$

From (11), for sufficiently large M, each maximum in the above is almost surely $\Theta\left(\log M^{2+\kappa}\right)$. Therefore, $\frac{1}{L}\sum_{\ell=1}^{L}\mathbb{E}_{\tilde{Z}_0,Z_0}\left|n^{\delta}\mathsf{diff}_{2,k,\ell}\right|^{2+\kappa}\overset{a.s.}{=}\Theta\left(L^{\delta}(\log M)^{\delta(2+\kappa)}\right)$. We have satisfied (10) since $M = L^b$ for some constant $b > 0$.

k=2, 3. Note that $\phi_{k,\ell}(x,y,\beta_{0_\ell}) \leq \frac{cnP_\ell}{\log M}$ for some constant c when $k = 2, 3$. The considering the difference in (9), we find the following upper bound:

$$\frac{1}{L}\sum_{\ell=1}^{L}\mathbb{E}_{\tilde{Z}_0,Z_0}\left|n^{\delta}\mathsf{diff}_{2,k,\ell}\right|^{2+\kappa} \leq 2n^{\delta(2+\kappa)}\left(\frac{cnP_\ell}{\log M}\right)^{2+\kappa} \overset{a.s.}{=} \Theta(n^{\delta(2+\kappa)}).$$

Therefore we have satisfied (10).

k=4. We first establish an upper bound for $\phi_{4,\ell}(x,y,z)$.

$$\begin{aligned}|\phi_{k,\ell}\left(a_0\bar{\tau}_0 Z_{0_\ell},\, b_0\bar{\tau}_0 Z_{0_\ell},\, \beta_{0_\ell}\right)|^{2+\kappa} = \frac{\left|b_0\bar{\tau}_0 Z_{0_\ell}^{*}[\eta_\ell^{r'}(\beta_0 - a_0\bar{\tau}_0 Z_0) - \beta_{0_\ell}]\right|^{2+\kappa}}{(\log M)^{2+\kappa}} &\leq \frac{c\left(\sqrt{nP_\ell}\right)^{2+\kappa}\max_{j\in sec(\ell)}|Z_{0_j}|^{2+\kappa}}{(\log M)^{2+\kappa}}\\ &\overset{a.s.}{=} \frac{\Theta(\sqrt{\log M}^{2+\kappa})\Theta(\sqrt{\log M}^{2+\kappa})}{(\log M)^{2+\kappa}}.\end{aligned} \tag{13}$$

The last equality is true for large enough M by (11). Therefore, $\frac{1}{L}\sum_{\ell=1}^{L}\mathbb{E}_{\tilde{Z}_0,Z_0}\left|n^{\delta}\mathsf{diff}_{2,k,\ell}\right|^{2+\kappa}\overset{a.s.}{=}\Theta(n^{\delta(2+\kappa)})$, and so we have satisfied (10).

We have satisfied (10) for each function $\phi_{k,\ell}(x,y,z)$ for $k = \{1,2,3,4\}$ and therefore result (8) holds. For the third and final step of the proof, we will provide the following result:

$$\lim n^{\delta}\left[\frac{1}{L}\sum_{\ell=1}^{L}\mathbb{E}_{Z_0}\left\{\phi_{k,\ell}\left(a_0\bar{\tau}_0 Z_{0_\ell}, b_0\bar{\tau}_0 Z_{0_\ell}, \beta_{0_\ell}\right)\right\} - \frac{1}{L}\sum_{\ell=1}^{L}\mathbb{E}_{(Z_0,\beta)}\left\{\phi_{k,\ell}\left(a_0\bar{\tau}_0 Z_{0_\ell}, b_0\bar{\tau}_0 Z_{0_\ell}, \beta_\ell\right)\right\}\right] \overset{a.s.}{=} 0.$$

For each function $\phi_{k,\ell}(x,y,z)$ with $k = \{1,2,3,4\}$ and each $\beta_0 \in \mathcal{B}_{M,L}$, we prove the following for each $\ell \in [L]$:

$$\mathbb{E}_{Z_0}\left\{\phi_{k,\ell}\left(a_0\bar{\tau}_0 Z_{0_\ell}, b_0\bar{\tau}_0 Z_{0_\ell}, \beta_{0_\ell}\right)\right\} = \mathbb{E}_{(Z_0,\beta)}\left\{\phi_{k,\ell}\left(a_0\bar{\tau}_0 Z_{0_\ell}, b_0\bar{\tau}_0 Z_{0_\ell}, \beta_\ell\right)\right\}.$$

The key is that the random variables $\left\{Z_{0_j}\right\}$ are i.i.d. across $j \in sec(\ell)$, and the position of the non-zero entry in β_ℓ is uniformly distributed across $j \in [M]$.

k=1. The result is trivially true since $\phi_{k,\ell}(x,y,z)$ doesn't depend on z.

k=2, 3. First consider the $k = 2$ function

$$\phi_{2,\ell}\left(a_0\bar{\tau}_0 Z_{0_\ell}, b_0\bar{\tau}_0 Z_{0_\ell}, \beta_{0_\ell}\right) = \frac{\|\eta_\ell^0(\beta - a_0\bar{\tau}_0 Z_0)\|^2}{\log M}.$$

Let $k \in sec(\ell)$ be the non-zero element of β_{0_ℓ}. Then for $i \in sec(\ell)$

$$\eta_i^0(\beta_0 - a_0\bar{\tau}_0 Z_0) = \sqrt{nP_\ell}\frac{\exp\left(\frac{nP_\ell}{\bar{\tau}_0^2}\cdot\mathbb{I}(i=k)\right)\exp\left(-\sqrt{nP_\ell}\,a_0\,Z_{0_i}\right)}{\exp\left(\frac{nP_\ell}{\bar{\tau}_0^2}\right)\exp\left(-\sqrt{nP_\ell}\,a_0\,Z_{0_k}\right)+\sum_{j\neq k}\exp\left(-\sqrt{nP_\ell}\,a_0\,Z_{0_j}\right)}.$$

Therefore,

$$\mathbb{E}_{Z_0}\left\{\phi_{k,\ell}\left(a_0\bar{\tau}_0 Z_{0_\ell}, b_0\bar{\tau}_0 Z_{0_\ell}, \beta_{0_\ell}\right)\right\}$$
$$= \frac{nP_\ell}{\log M}\mathbb{E}_{Z_0}\left[\frac{\exp\left(\frac{2nP_\ell}{\bar{\tau}_0^2}\right)\exp\left(-2\sqrt{nP_\ell}\, a_0\, Z_{0_k}\right) + \sum_{i\neq k}\exp\left(-2\sqrt{nP_\ell}\, a_0\, Z_{0_i}\right)}{\left(\exp\left(\frac{nP_\ell}{\bar{\tau}_0^2}\right)\exp\left(-\sqrt{nP_\ell}\, a_0\, Z_{0_k}\right) + \sum_{j\neq k}\exp\left(-\sqrt{nP_\ell}\, a_0\, Z_{0_j}\right)\right)^2}\right]. \tag{14}$$

The key observation is that the expectation on the RHS of (14) is the same regardless of whether the non-zero index i in β_{0_ℓ} is $1, 2, \ldots$, or M. This is because $\{Z_{0_j}\}$ is i.i.d. across $j \in \text{sec}(\ell)$. Hence,

$$\mathbb{E}_{(Z_0,\beta)}\left\{\phi_{k,\ell}\left(a_0\bar{\tau}_0 Z_{0_\ell}, b_0\bar{\tau}_0 Z_{0_\ell}, \beta_\ell\right)\right\} = \sum_{i=1}^{M}\frac{1}{M}\mathbb{E}_{Z_0}\left\{\phi_{k,\ell}\left(a_0\bar{\tau}_0 Z_{0_\ell}, b_0\bar{\tau}_0 Z_{0_\ell}, \beta_\ell\right)\middle|\,\text{Non-zero entry of } \beta_\ell \text{ is } i\right\}.$$

The above equals (14). The argument for the $k = 3$ function $\phi_{3,\ell}(x, y, z)$ similar.

k=4. The result can be shown in a manner similar to that used for the $k = 2$ function $\phi_{2,\ell}(x, y, z)$ shown above.

The existence of the limit of $\frac{1}{L}\sum_{\ell=1}^{L}\mathbb{E}_{(\breve{Z}_0,\beta)}[\phi_{k,\ell}(a_0\bar{\tau}_0\breve{Z}_{0_\ell}, b_0\bar{\tau}_0\breve{Z}_{0_\ell}, \beta_\ell)]$ for $k = 1$ follows from the law of large numbers; for $k = 2, 3, 4$, the limit is derived in Appendix A.4 in [1].

□

2 Step 4: Showing $\mathcal{H}_{t+1}$(b).(i) holds

We want to show that if [1, $\mathcal{B}_r$] and [1, $\mathcal{H}_s$] hold for $0 \leq r \leq t \leq T^*$ and $1 \leq s \leq t \leq T^*$ then (2) holds.

Proof. Consider the functions $\phi_{k,\ell}(x, y, z)$ for $k \in \{1, 2, 3, 4\}$ defined in (1). First note that the result of thsi Lemma, given in (2), is true for an additional group of functions defined as follows:

$$\phi_{5,\ell}(x, y, z) = \mathbb{E}_Z\left\{\phi_{k,\ell}(x + \sigma_Z Z, y + \sigma_Z Z, z)\right\},$$

where $k = \{1, 2, 3, 4\}$, random vector $Z \in \mathbb{R}^M$ is i.i.d. $\sim \mathcal{N}(0, 1)$ independent of x, y, z, and $\lim \sigma_Z \overset{a.s.}{=} \textit{constant}$. We do not prove the result explicitly for these functions since it follows by application of Jensen's inequality. In what follows we prove the result for generic $\phi_{k,\ell}(x, y, z)$ with $k \in \{1, 2, 3, 4\}$ and we state explicitly which k we refer to when it is important to the results.

From [1, Lemma 4] it follows,

$$\phi_{k,\ell}\left(\sum_{u=0}^{t} a_u h_\ell^{u+1}, \sum_{v=0}^{t} b_v h_\ell^{v+1}, \beta_{0_\ell}\right)\Bigg|_{\mathscr{S}_{t+1,t}}$$
$$\overset{d}{=} \phi_{k,\ell}\left(\sum_{u=0}^{t-1} a'_u h_\ell^{u+1} + a_t\bar{\tau}_t^{\perp} Z_{t_\ell} + a_t[\Delta_{t+1,t}]_\ell, \sum_{v=0}^{t-1} b'_v h_\ell^{v+1} + b_t\bar{\tau}_t^{\perp} Z_{t_\ell} + b_t[\Delta_{t+1,t}]_\ell, \beta_{0_\ell}\right),$$

where $a'_u = a_u$ and $b'_v = b_v$ for $0 \leq u, v \leq t-2$ and $a'_{t-1} = a_{t-1} + a_t(\bar{\tau}_t^2/\bar{\tau}_{t-1}^2)$ and $b'_{t-1} = b_{t-1} + b_t(\bar{\tau}_t^2/\bar{\tau}_{t-1}^2)$. Define

$$\begin{aligned}\mathsf{diff}_{1,k,\ell} := \phi_{k,\ell}&\left(\sum_{u=0}^{t-1} a'_u h_\ell^{u+1} + a_t\bar{\tau}_t^{\perp} Z_{t_\ell} + a_t[\Delta_{t+1,t}]_\ell, \sum_{v=0}^{t-1} b'_v h_\ell^{v+1} + b_t\bar{\tau}_t^{\perp} Z_{t_\ell} + b_t[\Delta_{t+1,t}]_\ell, \beta_{0_\ell}\right)\\ &- \phi_{k,\ell}\left(\sum_{u=0}^{t-1} a'_u h_\ell^{u+1} + a_t\bar{\tau}_t^{\perp} Z_{t_\ell}, \sum_{v=0}^{t-1} b'_v h_\ell^{v+1} + b_t\bar{\tau}_t^{\perp} Z_{t_\ell}, \beta_{0_\ell}\right).\end{aligned} \tag{15}$$

First, we show that the deviation term $\Delta_{t+1,t}$ can be dropped when considering the limit. Considering t fixed, for each function in (1) we first show the following.

$$\lim \frac{n^\delta}{L} \sum_{\ell=1}^{L} |\mathsf{diff}_{1,k,\ell}| \overset{a.s.}{=} 0. \tag{16}$$

for some $\delta > 0$. To prove the above we will supply upper bounds on the difference $\mathsf{diff}_{1,k,\ell}$ defined in (15) which approach 0 almost surely in the limit. To do so, we consider each k separately. The following two results are useful:

1. From [1, $\mathcal{H}_1$(a) - $\mathcal{H}_{t+1}$(a)], for each $\ell \in [L]$ and for constant $c_1 > 0$ not depending on N,
$$\max_{j \in sec(\ell)} |[\Delta_{t+1,t}]_j| \overset{a.s.}{=} \Theta(n^{-\delta'}\sqrt{\log M}), \quad \text{and} \quad \max_{j \in sec(\ell)} |h_j^{r+1}| \leq c_1 \sqrt{\log M} \quad 0 \leq r \leq t. \tag{17}$$

2. Result (17) along with Lemma 6 implies, for each $\ell \in [L]$ and for some constant $C > 0$, that
$$\max_{j \in sec(\ell)} \left| \sum_{u=0}^{t-1} a'_u h_j^{u+1} + a_t \bar{\tau}_t^\perp Z_{t_j} \right| \overset{a.s}{\leq} C\sqrt{\log M} \text{ for all sufficiently large } M. \tag{18}$$

$\mathbf{k = 1}$. Because of the $\Delta_{t+1,t}$ bounds given in (17), it follows from Lemma (A.1) for each $\ell \in [L]$,

$$|\mathsf{diff}_{1,k,\ell}| \overset{a.s.}{\leq} \frac{C\sqrt{\log M}}{n^{\delta'}} \left[\max_{j \in sec(\ell)} \left| \sum_{u=0}^{t-1} a'_u h_j^{u+1} + a_t \bar{\tau}_t^\perp Z_{t_j} \right| + \max_{j' \in sec(\ell)} \left| \sum_{v=0}^{t-1} b'_v h_j^{v+1} + b_t \bar{\tau}_t^\perp Z_{t_j} \right| + \frac{\sqrt{\log M}}{n^{\delta'}} \right],$$

Using bound (18) in the above, we find $\frac{n^\delta}{L}\sum_{\ell=1}^{L} |\mathsf{diff}_{1,k,\ell}| \overset{a.s.}{\leq} C' n^{\delta-\delta'} \log M$, which approaches 0 when $\delta < \delta'$ giving result (16).

$\mathbf{k = 2,3}$. Because of the $\Delta_{t+1,t}$ bounds given in (17), it follows from Lemma (A.2) for each $\ell \in [L]$, $|\mathsf{diff}_{1,k,\ell}| \overset{a.s.}{\leq} C n^{-\delta'} \log M$. Plugging this into (16),

$$\frac{n^\delta}{L} \sum_{\ell=1}^{L} |\mathsf{diff}_{1,k,\ell}| \overset{a.s.}{\leq} C' n^{\delta-\delta'} \log M.$$

The right-side of the above approaches 0 when $\delta < \delta'$.

$\mathbf{k = 4}$. Using the upper bound for $\phi_{4,\ell}(x,y,z)$ provided in (7), for constants $c, C > 0$,

$$(\log M)\, |\mathsf{diff}_{1,k,\ell}| \leq C\sqrt{\log M} \cdot \frac{c(\log M)^{3/2}}{n^{\delta'}} + 2\sqrt{nP_\ell} \cdot \Theta(n^{-\delta'}\sqrt{\log M}),$$

where we have used results (17), (18), and Lemma (A.2). Now using the above in (16), we find $\frac{n^\delta}{L}\sum_{\ell=1}^{L} |\mathsf{diff}_{1,k,\ell}| \overset{a.s.}{\leq} c' n^{\delta-\delta'} \log M + c'' n^{\delta-\delta'}$, which approaches 0 when $\delta < \delta'$.

In what follows we are justified in dropping the deviation terms $\Delta_{t+1,t}$. For the second step of the proof, we will appeal to the Strong Law for Triangular Arrays [1, Fact 2] which will show

$$\lim n^\delta \left[\frac{1}{L} \sum_{\ell=1}^{L} \phi_{k,\ell} \left(\sum_{u=0}^{t-1} a'_u h_\ell^{u+1} + a_t \bar{\tau}_t^\perp Z_{t_\ell}, \sum_{v=0}^{t-1} b'_v h_\ell^{v+1} + b_t \bar{\tau}_t^\perp Z_{t_\ell}, \beta_{0_\ell} \right) \right.$$
$$\left. - \frac{1}{L} \sum_{\ell=1}^{L} \mathbb{E}_{Z_t} \left\{ \phi_{k,\ell} \left(\sum_{u=0}^{t-1} a'_u h_\ell^{u+1} + a_t \bar{\tau}_t^\perp Z_{t_\ell}, \sum_{v=0}^{t-1} b'_v h_\ell^{v+1} + b_t \bar{\tau}_t^\perp Z_{t_\ell}, \beta_{0_\ell} \right) \right\} \right] \overset{a.s.}{=} 0. \tag{19}$$

Let $\tilde{Z}_t$ be an independent copy of Z_t and define for each $k = \{1, 2, 3, 4\}$,

$$\begin{aligned}\mathsf{diff}_{2,k,\ell} := \phi_{k,\ell}\left(\sum_{u=0}^{t-1} a'_u h_\ell^{u+1} + a_t \bar{\tau}_t^\perp \tilde{Z}_{t_\ell}, \sum_{v=0}^{t-1} b'_v h_\ell^{v+1} + b_t \bar{\tau}_t^\perp \tilde{Z}_{t_\ell}, \beta_{0_\ell}\right) \\ - \phi_{k,\ell}\left(\sum_{u=0}^{t-1} a'_u h_\ell^{u+1} + a_t \bar{\tau}_t^\perp Z_{t_\ell}, \sum_{v=0}^{t-1} b'_v h_\ell^{v+1} + b_t \bar{\tau}_t^\perp Z_{t_\ell}, \beta_{0_\ell}\right).\end{aligned} \tag{20}$$

In order to use the Strong Law for Triangular Arrays [1, Fact 2] to get result (19) we will prove the following for each function in (1).

$$\frac{1}{L}\sum_{\ell=1}^{L} \mathbb{E}_{\tilde{Z}_t, Z_t}\left|n^\delta \mathsf{diff}_{2,k,\ell}\right|^{2+\kappa} \leq cL^{\kappa/2}, \tag{21}$$

for some $0 \leq \kappa \leq 1$ and c some constant. Note that the exact requirement of [1, Fact 2] is met by an application of Jensen's Inequality. To prove (21), we consider each k separately. The following result, which follows from (6) for sufficiently large M, will be useful. Let $Z \in \mathbb{R}^M$ be a vector of i.i.d. standard Gaussian random variables,

$$\left(\max_{j \in sec(\ell)}\left|\sum_{u=0}^{t-1} a'_u h_j^{u+1} + a_t \bar{\tau}_t^\perp Z_j\right|\right)^{2+\kappa} = \max_{j \in sec(\ell)}\left|\sum_{u=0}^{t-1} a'_u h_j^{u+1} + a_t \bar{\tau}_t^\perp Z_j\right|^{2+\kappa} \overset{a.s}{=} \Theta\left(\sqrt{\log M}^{2+\kappa}\right). \tag{22}$$

k=1. As in (12), first we upper bound the difference in (20) as follows:

$$\begin{aligned}|\mathsf{diff}_{2,k,\ell}|^{2+\kappa} \leq \max_{j \in sec(\ell)}\left|\sum_{u=0}^{t-1} a'_u h_\ell^{u+1} + a_t \bar{\tau}_t^\perp \tilde{Z}_{t_\ell}\right|^{2+\kappa} \max_{j' \in sec(\ell)}\left|\sum_{v=0}^{t-1} b'_v h_\ell^{v+1} + b_t \bar{\tau}_t^\perp \tilde{Z}_{t_\ell}\right|^{2+\kappa} \\ + \max_{j \in sec(\ell)}\left|\sum_{u=0}^{t-1} a'_u h_\ell^{u+1} + a_t \bar{\tau}_t^\perp Z_{t_\ell}\right|^{2+\kappa} \max_{j' \in sec(\ell)}\left|\sum_{v=0}^{t-1} b'_v h_\ell^{v+1} + b_t \bar{\tau}_t^\perp Z_{t_\ell}\right|^{2+\kappa}.\end{aligned}$$

From (22), for sufficiently large M, each maximum in the above is almost surely $\Theta\left(\sqrt{\log M}^{2+\kappa}\right)$. Therefore, $\frac{1}{L}\sum_{\ell=1}^{L} \mathbb{E}_{\tilde{Z}_0, Z_0}\left|n^\delta \mathsf{diff}_{2,k,\ell}\right|^{2+\kappa} \overset{a.s.}{=} \Theta\left(L^\delta (\log M)^{\delta(2+\kappa)}\right)$. We have satisfied (21) since $M = L^b$ for some constant $b > 0$.

k=2,3. Note that $\phi_{k,\ell}(x, y, \beta_{0_\ell}) \leq \frac{cnP_\ell}{\log M}$ for some constant c when $k = 2, 3$. The considering the difference in (20), we find the following upper bound:

$$\frac{1}{L}\sum_{\ell=1}^{L} \mathbb{E}_{\tilde{Z}_t, Z_t}\left|n^\delta \mathsf{diff}_{2,k,\ell}\right|^{2+\kappa} \leq 2n^{\delta(2+\kappa)}\left(\frac{cnP_\ell}{\log M}\right)^{2+\kappa} \overset{a.s.}{=} \Theta(n^{\delta(2+\kappa)}).$$

Therefore we have satisfied (21).

k=4. As in (13) we first establish an upper bound for $\phi_{4,\ell}(x,y,z)$,

$$\left|\phi_{k,\ell}\left(\sum_{u=0}^{t-1} a'_u h_\ell^{u+1} + a_t \bar{\tau}_t^\perp Z_{t_\ell}, \sum_{v=0}^{t-1} b'_v h_\ell^{v+1} + b_t \bar{\tau}_t^\perp Z_{t_\ell}, \beta_{0_\ell}\right)\right|^{2+\kappa}$$
$$= \frac{\left|(\sum_{v=0}^{t-1} b'_v h_\ell^{v+1} + b_t \bar{\tau}_t^\perp Z_{t_\ell})^* [\eta_\ell^{r'}(\beta_0 - \sum_{u=0}^{t-1} a'_u h^{u+1} - a_t \bar{\tau}_t^\perp Z_t) - \beta_{0_\ell}]\right|^{2+\kappa}}{(\log M)^{2+\kappa}}$$
$$\leq \frac{\left(2\sqrt{nP_\ell}\right)^{2+\kappa} \max_{j \in sec(\ell)} \left|\sum_{v=0}^{t-1} b'_v h_\ell^{v+1} + b_t \bar{\tau}_t^\perp Z_{t_\ell}\right|^{2+\kappa}}{(\log M)^{2+\kappa}} \overset{a.s.}{=} \frac{\Theta(\sqrt{\log M}^{2+\kappa})\Theta(\sqrt{\log M}^{2+\kappa})}{(\log M)^{2+\kappa}}.$$

The last equality is true, for sufficiently large M, by (22). Therefore, $\frac{1}{L}\sum_{\ell=1}^{L} \mathbb{E}_{\tilde{Z}_t, Z_t} \left|n^\delta \mathsf{diff}_{2,k,\ell}\right|^{2+\kappa} \leq 2n^{\delta(2+\kappa)} \left(\frac{cnP_\ell}{\log M}\right)^{2+\kappa} \overset{a.s.}{=} \Theta(n^{\delta(2+\kappa)})$. We have satisfied (21) for $\mathsf{Class\ 2}$ functions.

We have satisfied (21) for each function $\phi_{k,\ell}(x,y,z)$ for $k = \{1,2,3,4\}$ and therefore result (19) holds. Considering result (19), define new functions $\phi_{k,\ell}^{NEW}$ for $k = \{1,2,3,4\}$ as

$$\phi_{k,\ell}^{NEW}\left(\sum_{u=0}^{t-1} a'_u h_\ell^{u+1}, \sum_{v=0}^{t-1} b'_v h_\ell^{v+1}, \beta_{0_\ell}\right) := \mathbb{E}_{Z_t}\left\{\phi_{k,\ell}\left(\sum_{u=0}^{t-1} a'_u h_\ell^{u+1} + a_t \bar{\tau}_t^\perp Z_{t_\ell}, \sum_{v=0}^{t-1} b'_v h_\ell^{v+1} + b_t \bar{\tau}_t^\perp Z_{t_\ell}, \beta_{0_\ell}\right)\right\}.$$

Using Jensen's inequality, it can be shown that the induction hypothesis [1, $\mathcal{H}_t$(b)] holds for the function $\phi_{k,\ell}^{NEW}$ whenever $\mathcal{H}_t(b)$ holds for the function $\phi_{k,\ell}$ inside the expectation. Therefore,

$$\lim n^\delta \left[\frac{1}{L}\sum_{\ell=1}^{L} \mathbb{E}_{Z_t}\left\{\phi_{k,\ell}\left(\sum_{u=0}^{t-1} a'_u h_\ell^{u+1} + a_t \bar{\tau}_t^\perp Z_{t_\ell}, \sum_{v=0}^{t-1} b'_v h_\ell^{v+1} + b_t \bar{\tau}_t^\perp Z_{t_\ell}, \beta_{0_\ell}\right)\right\}\right.$$
$$\left. - \frac{1}{L}\sum_{\ell=1}^{L} \mathbb{E}\mathbb{E}_{Z_t}\left\{\phi_{k,\ell}\left(\sum_{u=0}^{t-1} a'_u \bar{\tau}_u \breve{Z}_{u_\ell} + a_t \bar{\tau}_t^\perp Z_{t_\ell}, \sum_{v=0}^{t-1} b'_v \bar{\tau}_v \breve{Z}_{v_\ell} + b_t \bar{\tau}_t^\perp Z_{t_\ell}, \beta_\ell\right)\right\}\right] \overset{a.s.}{=} 0.$$

To complete the proof we show that

$$\mathbb{E}\mathbb{E}_{Z_t}\{\phi_{k,\ell}(\sum_{u=0}^{t-1} a'_u \bar{\tau}_u \breve{Z}_u + a_t \bar{\tau}_t^\perp Z_t, \sum_{v=0}^{t-1} b'_v \bar{\tau}_v \breve{Z}_v + b_t \bar{\tau}_t^\perp Z_t, \beta_\ell)\} = \mathbb{E}\{\phi_{k,\ell}(\sum_{u=0}^{t} a_u \bar{\tau}_u \breve{Z}_u, \sum_{v=0}^{t} b_v \bar{\tau}_v \breve{Z}_v, \beta_\ell)\}.$$

Recall $a'_{t-1} = a'_{t-1} = a_{t-1} + a_t(\bar{\tau}_t^2/\bar{\tau}_{t-1}^2)$ and $b'_{t-1} = b_{t-1} + b_t(\bar{\tau}_t^2/\bar{\tau}_{t-1}^2)$. Then to prove the above we will show that $(\bar{\tau}_t^2/\bar{\tau}_{t-1})\breve{Z}_{t-1} + \bar{\tau}_t^\perp Z_t \overset{d}{=} \bar{\tau}_t \breve{Z}_t$ where $\bar{\tau}_r \bar{\tau}_t \mathbb{E}[\breve{Z}_r \breve{Z}_t] = \bar{\tau}_t^2$ for $0 \leq r \leq t-1$. Note that $\left((\bar{\tau}_t^2/\bar{\tau}_{t-1})\breve{Z}_{t-1} + \bar{\tau}_t^\perp Z_t\right)$ is Gaussian with variance equal to $(\bar{\tau}_t^2/\bar{\tau}_{t-1})^2 + (\bar{\tau}_t^\perp)^2 = \bar{\tau}_t^2$ using the definition of $\bar{\tau}_t^\perp$. This follows since $\breve{Z}_{t-1}$ and Z_t are independent. Finally, for $0 \leq r \leq t-1$

$$\mathbb{E}\left\{\bar{\tau}_r \breve{Z}_r \left((\bar{\tau}_t^2/\bar{\tau}_{t-1})\breve{Z}_{t-1} + \bar{\tau}_t^\perp Z_t\right)\right\} = (\bar{\tau}_t^2/\bar{\tau}_{t-1}^2)\bar{\tau}_r \bar{\tau}_{t-1} \mathbb{E}[\breve{Z}_r \breve{Z}_{t-1}] = \bar{\tau}_t^2.$$

The existence of the limit of $\mathbb{E}\{\phi_{k,\ell}(\sum_{u=0}^{t} a_u \bar{\tau}_u \breve{Z}_u, \sum_{v=0}^{t} b_v \bar{\tau}_v \breve{Z}_v, \beta_\ell)\}$ for $k = 1$ follows from the law of large numbers; for $k = 2,3,4$, the existence of the limit follows from Appendix A.4 in [1]. This completes the proof.

□

A Appendix

Lemma A.1 ($k = 1$ Function Bound)**.** *We consider the function $\phi_{1,\ell} : \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}$ defined as $\phi_{1,\ell}(x, y, z) = x^* y/M$. If $\max_{j\in[M]}|\Delta_{x_j}|$ and $\max_{j\in[M]}|\Delta_{y_j}|$ are both almost surely $\Theta(n^{-\delta}\sqrt{\log M})$ for some $\delta > 0$, the following holds for some constant $C > 0$:*

$$|\phi_h(x, y, z) - \phi_h(x + \Delta_x, y + \Delta_y, z)| \overset{a.s.}{\leq} \frac{C\sqrt{\log M}}{n^\delta}\left[\max_{j\in[M]}|x_j| + \max_{j'\in[M]}|y_{j'}| + \frac{\sqrt{\log M}}{n^\delta}\right]. \tag{23}$$

Proof.

$$\begin{aligned}|\phi_h(x, y, z) - \phi_h(x + \Delta_x, y + \Delta_y, z)| &= \frac{1}{M}\left|x^* y - (x + \Delta_x)^*(y + \Delta_y)\right| \\ &\leq \frac{1}{M}\left|x^*\Delta_y\right| + \frac{1}{M}\left|\Delta_x^*(y + \Delta_y)\right| \\ &\leq \max_{j\in[M]}|\Delta_{y_j}| \max_{i\in[M]}|x_i| + \max_{j\in[M]}|\Delta_{x_j}|\left(\max_{i\in[M]}|y_i| + \max_{i'\in[M]}|\Delta_{y_i'}|\right).\end{aligned}$$

The result follows since $\max_{j\in[M]}|\Delta_{x_j}|$ and $\max_{j\in[M]}|\Delta_{y_j}|$ are both almost surely $\Theta(n^{-\delta}\sqrt{\log M})$. ☐

Lemma A.2 ($k = 2, 3$ Function Bound)**.** *We consider the functions $\phi_{k,\ell} : \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}$ for $k = 2, 3$ defined as*

$$\begin{aligned}&\phi_{2,\ell}(x, y, z) := \|\eta^r_\ell(z - x)\|^2/\log M, && 0 \leq r \leq t, \\ &\phi_{3,\ell}(x, y, z) := [\eta^r_\ell(z - x) - z]^*[\eta^s_\ell(z - y) - z]/\log M, && -1 \leq r \leq s \leq t,\end{aligned} \tag{24}$$

If $\max_{j\in[M]}|z_j| = \sqrt{nP_\ell}$ and $\max_{j\in[M]}|\Delta_{x_j}|$ and $\max_{j\in[M]}|\Delta_{y_j}|$ are both almost surely $\Theta(n^{-\delta}\sqrt{\log M})$ for some $\delta > 0$ and each $\ell \in [L]$, the following holds for some constant $C > 0$, for $k = 2, 3$:

$$|\phi_{k,\ell}(x, y, z) - \phi_{k,\ell}(x + \Delta_x, y + \Delta_y, z)| \overset{a.s.}{\leq} Cn^{-\delta}\log M.$$

Furthermore for $0 \leq u \leq t$,

$$\sum_{i=1}^{M}|\eta^u_i(z - x - \Delta_x) - \eta^u_i(z - x)| \overset{a.s.}{\leq} Cn^{-\delta}(\log M)^{3/2}. \tag{25}$$

Proof. In the following assume $\ell \in [L]$ is fixed, and therefore for $i \in [M]$ (and $i \in sec(\ell)$) we let,

$$\eta^r_i(v) = \sqrt{nP_\ell}\exp\left\{\frac{v_i\sqrt{nP_\ell}}{\bar{\tau}^2_r}\right\}\left[\sum_{j=1}^{M}\exp\left\{\frac{v_j\sqrt{nP_\ell}}{\bar{\tau}^2_r}\right\}\right]^{-1}.$$

Define the following:

$$F^1_{c,\Delta,\bar{\tau}^2} := \exp\left\{\frac{c\sqrt{nP_\ell}}{\bar{\tau}^2}\max_{j\in[M]}|\Delta_j|\right\} - 1, \text{ and } F^2_{c,\Delta,\bar{\tau}^2} := 1 - \exp\left\{-\frac{c\sqrt{nP_\ell}}{\bar{\tau}^2}\max_{j\in[M]}|\Delta_j|\right\}. \tag{26}$$

In what follows we upper bound both functions $k = 2, 3$ using the above definition as follows for some constantc $C, c > 0$:

$$|\phi_{k,\ell}(x, y, z) - \phi_{k,\ell}(x + \Delta_x, y + \Delta_y, z)| \overset{a.s.}{\leq} \frac{CnP_\ell}{\log M}\max\left(F^1_{c,\Delta_x,\bar{\tau}^2_r}, F^1_{c,\Delta_y,\bar{\tau}^2_s}, F^2_{c,\Delta_x,\bar{\tau}^2_r}, F^2_{c,\Delta_y,\bar{\tau}^2_s}\right). \tag{27}$$

Then a Taylor expansion of e^x can be used to show that each of $F^1_{c,\Delta_x,\bar{\tau}_r^2}, F^1_{c,\Delta_y,\bar{\tau}_s^2}, F^2_{c,\Delta_x,\bar{\tau}_r^2}, F^2_{c,\Delta_y,\bar{\tau}_s^2}$ can be upper bounded almost surely by $c' n^{-\delta} \log M$ whenever $\max_{j\in[M]}|\Delta_{x_j}|$ and $\max_{j\in[M]}|\Delta_{y_j}|$ are both almost surely $\Theta(n^{-\delta}\sqrt{\log M})$. Therefore

$$\max(F^1_{c,\Delta_x,\bar{\tau}_r^2}, F^1_{c,\Delta_y,\bar{\tau}_s^2}, F^2_{c,\Delta_x,\bar{\tau}_r^2}, F^2_{c,\Delta_y,\bar{\tau}_s^2}) \overset{a.s.}{\leq} C' n^{-\delta} \log M, \tag{28}$$

for constant $C' > 0$, which along with the bound in (27) provides the desired result. What remains is to prove (27) for $k = 2, 3$.

Now to complete the proof we show that upper bound (27) for both functions $k = 2, 3$.

k=2 First note,

$$\phi_{2,\ell}(x, y, z) = \frac{\|\eta^r(z-x)\|^2}{\log M} = \frac{nP_\ell}{\log M} \frac{\sum_{i=1}^M \exp\left\{2(z_i - x_i)\frac{\sqrt{nP_\ell}}{\bar{\tau}_r^2}\right\}}{\left(\sum_{j=1}^M \exp\left\{(z_j - x_j)\frac{\sqrt{nP_\ell}}{\bar{\tau}_r^2}\right\}\right)^2}.$$

From the above we can write

$$\begin{aligned}
\frac{\log M}{nP_\ell}\phi_{2,\ell}(x, y, z) &= \frac{\sum_{i=1}^M \exp\left\{2(z_i - x_i - \Delta_{x_i})\frac{\sqrt{nP_\ell}}{\bar{\tau}_r^2}\right\}\exp\left\{2\Delta_{x_i}\frac{\sqrt{nP_\ell}}{\bar{\tau}_r^2}\right\}}{\left(\sum_{j=1}^M \exp\left\{(z_j - x_j - \Delta_{x_j})\frac{\sqrt{nP_\ell}}{\bar{\tau}_r^2}\right\}\exp\left\{\Delta_{x_j}\frac{\sqrt{nP_\ell}}{\bar{\tau}_r^2}\right\}\right)^2} \\
&\leq \frac{\sum_{i=1}^M \exp\left\{2(z_i - x_i - \Delta_{x_i})\frac{\sqrt{nP_\ell}}{\bar{\tau}_r^2}\right\}\exp\left\{2\left(\max_{j'\in[M]}|\Delta_{x'_j}|\right)\frac{\sqrt{nP_\ell}}{\bar{\tau}_r^2}\right\}}{\exp\left\{-2\left(\max_{j'\in[M]}|\Delta_{x'_j}|\right)\frac{\sqrt{nP_\ell}}{\bar{\tau}_r^2}\right\}\left(\sum_{j=1}^M \exp\left\{(z_j - x_j - \Delta_{x_j})\frac{\sqrt{nP_\ell}}{\bar{\tau}_r^2}\right\}\right)^2},
\end{aligned} \tag{29}$$

and similarly

$$\frac{\log M}{nP_\ell}\phi_{2,\ell}(x, y, z) \geq \frac{\sum_{i=1}^M \exp\left\{2(z_i - x_i - \Delta_{x_i})\frac{\sqrt{nP_\ell}}{\bar{\tau}_r^2}\right\}\exp\left\{-2\left(\max_{j'\in[M]}|\Delta_{x'_j}|\right)\frac{\sqrt{nP_\ell}}{\bar{\tau}_r^2}\right\}}{\exp\left\{2\left(\max_{j'\in[M]}|\Delta_{x'_j}|\right)\frac{\sqrt{nP_\ell}}{\bar{\tau}_r^2}\right\}\left(\sum_{j=1}^M \exp\left\{(z_j - x_j - \Delta_{x_j})\frac{\sqrt{nP_\ell}}{\bar{\tau}_r^2}\right\}\right)^2}. \tag{30}$$

Putting (29) and (30) together we see that

$$\exp\left\{-\frac{4\sqrt{nP_\ell}}{\bar{\tau}_r^2}\max_{j'\in[M]}|\Delta_{x'_j}|\right\} \leq \frac{\phi_{2,\ell}(x, y, z)}{\phi_{2,\ell}(x+\Delta_x, y+\Delta_y, z)} \leq \exp\left\{\frac{4\sqrt{nP_\ell}}{\bar{\tau}_r^2}\max_{j'\in[M]}|\Delta_{x'_j}|\right\}, \tag{31}$$

and so we find the desired result (27) for k=2:

$$\begin{aligned}
|\phi_{2,\ell}(x, y, z) - \phi_{2,\ell}(x+\Delta_x, y+\Delta_y, z)| &\leq \phi_{2,\ell}(x+\Delta_x, y+\Delta_y, z)\max\left(F^1_{4,\Delta_x,\bar{\tau}_r^2}, F^2_{4,\Delta_x,\bar{\tau}_r^2}\right) \\
&\leq \frac{nP_\ell}{\log M}\max\left(F^1_{4,\Delta_x,\bar{\tau}_r^2}, F^2_{4,\Delta_x,\bar{\tau}_r^2}\right).
\end{aligned} \tag{32}$$

k=3 First note,

$$\begin{aligned}
&(\log M)\left[\phi_{3,\ell}(x, y, z) - \phi_{3,\ell}(x+\Delta_x, y+\Delta_y, z)\right] \\
&\quad = \eta^r(z-x)^*\eta^s(z-y) - \eta^r(z-x-\Delta_x)^*\eta^s(z-y-\Delta_y) \\
&\qquad - z^*\left[\eta^r(z-x) - \eta^r(z-x-\Delta_x)\right] - z^*\left[\eta^s(z-y) - \eta^s(z-y-\Delta_y)\right].
\end{aligned} \tag{33}$$

We again suppress the explicit notation for the dependence on section ℓ. Then to prove result (27) we prove the following two upper bounds.

1. For $0 \leq r \leq t$,

$$|z^* [\eta^r(z-x) - \eta^r(z-x-\Delta_x)]| \leq nP_\ell \max\left(F^1_{2,\Delta_x,\bar{\tau}_r^2}, F^2_{2,\Delta_x,\bar{\tau}_r^2}\right). \tag{34}$$

2. For $0 \leq r \leq s \leq t$,

$$\begin{aligned} &|\eta^r(z-x)^*\eta^s(z-y) - \eta^r(z-x-\Delta_x)^*\eta^s(x-y-\Delta_y)| \\ &\qquad \leq 2nP_\ell \max\left(F^1_{2,\Delta_x,\bar{\tau}_r^2}, F^1_{2,\Delta_y,\bar{\tau}_s^2}, F^2_{2,\Delta_x,\bar{\tau}_r^2}, F^2_{2,\Delta_y,\bar{\tau}_s^2}\right). \end{aligned} \tag{35}$$

Both bounds are trivial for the $r = -1$ case. Using (34) and (35) in (33) we get the desired result (27). First consider (34):

$$\begin{aligned} |z^* [\eta^r(z-x-\Delta_x) - \eta^r(z-x)]| &\leq \max_{j\in[M]}|z_j| \sum_{i=1}^{M} |\eta^r_i(z-x-\Delta_x) - \eta^r_i(z-x)| \\ &= \sqrt{nP_\ell} \sum_{i=1}^{M} |\eta^r_i(z-x-\Delta_x) - \eta^r_i(z-x)|. \end{aligned} \tag{36}$$

Next consider (35).

$$\begin{aligned} &|\eta^r(z-x)^*\eta^s(z-y) - \eta^r(z-x-\Delta_x)^*\eta^s(z-y-\Delta_y)| \\ &\leq |\eta^r(z-x)^*[\eta^s(z-y) - \eta^s(z-y-\Delta_y)]| + |[\eta^r(z-x) - \eta^r(z-x-\Delta_x)]^*\eta^s(z-y-\Delta_y)| \\ &\leq \sqrt{nP_\ell} \sum_{i=1}^{M} |\eta^s_i(z-y) - \eta^s_i(z-y-\Delta_y)| + \sqrt{nP_\ell} \sum_{i=1}^{M} |\eta^r_i(z-x) - \eta^r_i(z-x-\Delta_x)|. \end{aligned} \tag{37}$$

Now considering (36) and (37), to get the results given in (34) and (35) we demonstrate the following for $0 \leq u \leq t$,

$$\sum_{i=1}^{M} |\eta^u_i(z-x-\Delta_x) - \eta^u_i(z-x)| \leq \sqrt{nP_\ell} \max(F^1_{2,\Delta_x,\bar{\tau}_r^2}, F^2_{2,\Delta_x,\bar{\tau}_r^2}). \tag{38}$$

Note that the above leads to result (25) using (28). We now prove (38). Using a strategy similar to that in in (31),

$$\exp\left\{-\frac{2\sqrt{nP_\ell}}{\bar{\tau}_r^2} \max_{j'\in[M]}|\Delta_{x'_j}|\right\} \leq \frac{\eta^u_i(z-x)}{\eta^u_i(z-x-\Delta_x)} \leq \exp\left\{\frac{2\sqrt{nP_\ell}}{\bar{\tau}_r^2} \max_{j'\in[M]}|\Delta_{x'_j}|\right\}.$$

Result (38) follows from the above as in (32). □

References

[1] C. Rush, A. Greig, and R. Venkataramanan, "Capacity-achieving sparse superposition codes via approximate message passing decoding," *IEEE Trans. Inf. Theory*, 2017. Online: http://arxiv.org/abs/1501.05892.