Extended Proof of Steps 2(b) and 4(b)

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This document serves as an extended proof $H_1$ and $H_{t+1}$ of Lemma 5, part (b).(i) from Capacity-achieving Sparse Superposition Codes via Approximate Message Passing Decoding [1]. References to [1] and the proof of Lemma 5 within, will be made throughout this document.

**Lemma 5 (b)(i)** We will show that the following statement holds for $0 \leq t \leq T^*$, where $T^* = \left\lceil \frac{2C}{\log(C/R)} \right\rceil$ assumed to be less than $n$. Consider the following functions defined on $\mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^M \rightarrow \mathbb{R}$. For $x, y, z \in \mathbb{R}^M$ and $\ell \in [L]$, let

\[
\begin{align*}
\phi_{1,\ell}(x, y, z) &:= x^*y/M, \\
\phi_{2,\ell}(x, y, z) &:= ||\eta^\ell_\ell(z - x)||^2/\log M, \\
\phi_{3,\ell}(x, y, z) &:= [\eta^\ell_\ell(z - x) - z][\eta^\ell_\ell(z - y) - z]/\log M, \\
\phi_{4,\ell}(x, y, z) &:= y^*[\eta^\ell_\ell(z - x) - z]/\log M, \\
\end{align*}
\]

where for $r \geq 0$, $\eta^\ell_r(\cdot)$ is the restriction of $\eta^\ell_r$ to section $\ell$, i.e., for $x \in \mathbb{R}^M$, $\eta^\ell_{r,i}(x) := \sqrt{n}\exp\left(\frac{x_i\sqrt{nP_\ell}}{\tau^\ell_r}\right)\sum_{j=1}^M \exp\left(\frac{x_j\sqrt{nP_\ell}}{\tau^\ell_r}\right)$, $i = 1, \ldots, M$.

(Also, $\eta^\ell_{r,i}(\cdot) := 0$ for $i \in [M]$.) Then, for $k \in \{1, 2, 3, 4\}$ and arbitrary constants $(a_0, \ldots, a_t, b_0, \ldots, b_t)$, we have

\[
\lim n^\delta \left| \frac{1}{L} \sum_{\ell=1}^L \phi_{k,\ell} \left( \sum_{r=0}^t a_r h_{\ell,r}^{s+1}, \sum_{s=0}^t b_s h_{\ell,s}^{s+1}, \beta_{\ell} \right) - c_k \right| = 0,
\]

where

\[
c_k := \lim n^\delta \left| \frac{1}{L} \sum_{\ell=1}^L \mathbb{E} \left[ \phi_{k,\ell} \left( \sum_{r=0}^t a_r \tau_{\ell,r}, \sum_{s=0}^t b_s \tau_{\ell,s}, \beta_{\ell} \right) \right] \right|
\]

Here $\tilde{Z}_0, \ldots, \tilde{Z}_t$ are length-$N$ Gaussian random vectors independent of $\beta$, with $\tilde{Z}_{\ell}$ denoting the $\ell$th section of $\tilde{Z}_r$. For $0 \leq s \leq t$, $\{\tilde{Z}_{s,j}\}_{j \in [N]}$ are i.i.d. $\mathcal{N}(0, 1)$, and for each $i \in [N]$, $(\tilde{Z}_{0,i}, \ldots, \tilde{Z}_{t,i})$ are jointly Gaussian with $\mathbb{E}[\tau_{\ell,r} \tilde{Z}_{\ell,r}] = \tau_{\ell,r}^2$ for $0 \leq r \leq t$. Both limits in (2) exists and are finite for each $\phi_{k,\ell}$ in (1).

In the proof, we will use Lemmas A.1 and A.2 which are stated and proved in the Appendix.

1 Step 2: Showing $H_1(b).(i)$ holds

We will show that result (2) holds when $t = 0$. 
Proof. Consider the functions $\phi_k,\ell(x,y,z)$ for $k \in \{1,2,3,4\}$ defined in (1). First note that the result of this Lemma, given in (2), is true for an additional group of functions defined as follows:

$$\phi_{5,\ell}(x,y,z) = \mathbb{E}_Z \{ \phi_{k,\ell}(x + \sigma Z y + \sigma Z z) \},$$

where $k = \{1,2,3,4\}$, random vector $Z \in \mathbb{R}^n$ is i.i.d. $\sim \mathcal{N}(0,1)$ independent of $x,y,z$, and $\lim \sigma Z = \text{constant}$. We do not prove the result explicitly for these functions since it follows by application of Jensen's Inequality. In what follows we prove the result for generic $\phi_{k,\ell}(x,y,z)$ with $k \in \{1,2,3,4\}$ and we state explicitly which $k$ we refer to when it is important to the results.

From [1] Lemma 4 it follows,

$$\phi_k,\ell(a_0 h_1^a, b_0 h_1^b, \beta_0) |_{\mathcal{H}_a} \overset{d}{=} \phi_k,\ell (a_0 \tau_0 Z_0 + a_0 [\Delta_{1,0}]_\ell, b_0 \tau_0 Z_0 + b_0 [\Delta_{1,0}]_\ell, \beta_0).$$

For the first step, we show that the deviation term $\Delta_{1,0}$ can be dropped when considering the limit. Define

$$\text{diff}_{1,k,\ell} := \phi_k,\ell(a_0 \tau_0 Z_0 + a_0 [\Delta_{1,0}]_\ell, b_0 \tau_0 Z_0 + b_0 [\Delta_{1,0}]_\ell, \beta_0) - \phi_k,\ell(a_0 \tau_0 Z_0, b_0 \tau_0 Z_0, \beta_0).$$

Considering $t$ fixed, for each function in (1) we first show the following.

$$\lim \frac{n^\delta}{L} \sum_{\ell=1}^L \text{diff}_{1,k,\ell} \overset{a.s.}{=} 0. \tag{4}$$

To prove the above we will supply upper bounds on the difference $\text{diff}_{1,k,\ell}$ defined in (3) which approach 0 almost surely in the limit. To do so, we consider each $k$ separately. The following two results are useful:

1. From [1] $\mathcal{H}_1(a)$, for each $\ell \in [L],$

$$\max_{j \in \text{sec}(\ell)} | |\Delta_{1,0}|_j \overset{a.s.}{=} \Theta(n^{-\delta'} \sqrt{\log M}), \quad \text{and} \quad \max_{j \in \text{sec}(\ell)} |h^j_1| \leq c_1 \sqrt{\log M}, \tag{5}$$

where $c_1 > 0$ is a constant not depending on $N$.

2. The following is due to [1] Fact 7. For $Z_1, Z_2, \ldots$ i.i.d. $\sim \mathcal{N}(0,1)$, with probability 1 we have

$$\max_{j \in [M]} |Z_j| \leq \sqrt{2K \log M} \text{ for all sufficiently large } M. \tag{6}$$

$k = 1$. Because of the $\Delta_{1,0}$ bounds given in (5), it follows from Lemma (A.1) for each $\ell \in [L],$

$$|\text{diff}_{1,k,\ell} \overset{a.s.}{\leq} C \frac{\sqrt{\log M}}{n^{\delta'}} \left[ \max_{j \in \text{sec}(\ell)} \left| a_0 \tau_0 \max_{j \in \text{sec}(\ell)} |Z_0| + \max_{j' \in \text{sec}(\ell)} |Z_0| \right| \right],$$

for some $\delta' > 0$. Considering (6) and the above, $n^\delta \sum_{\ell=1}^L \text{diff}_{1,k,\ell} \overset{a.s.}{\leq} C' n^{\delta - \delta'} \log M$, which approaches 0 when $\delta < \delta'$.

$k = 2, 3$. Because of the $\Delta_{1,0}$ bounds given in (7), it follows from Lemma (A.2) for each $\ell \in [L],$

$$|\text{diff}_{1,k,\ell} \overset{a.s.}{\leq} C \log M n^{-\delta'},$$

for some $\delta' > 0$. Now plugging this into (4) we establish the following upper bound:

$$\frac{n^\delta}{L} \sum_{\ell=1}^L \text{diff}_{1,k,\ell} \overset{a.s.}{\leq} \frac{n^\delta}{L} \sum_{\ell=1}^L \frac{C \log M}{n^{\delta'}} \leq C n^{\delta - \delta'} \log M.$$
The right-side of the above approaches 0 when $\delta < \delta'$.

**k = 4.** We first establish an upper bound for $\phi_{4,\ell}(x, y, z)$.

\[
(\log M) |\phi_{k,\ell}(x, y, z) - \phi_{4,\ell}(x + \Delta x, y + \Delta y, z)|
\]

\[
= |y^* [\eta_k^r(z - x) - z] - (y + \Delta y)^* [\eta_k^r(z - x - \Delta x) - z]|
\]

\[
\leq |y^* [\eta_k^r(z - x) - \eta_k^r(z - x - \Delta x)]| + |\Delta_y^*[\eta_k^r(z - x - \Delta x) - z]|
\]

\[
\leq c (\log M)^{3/2} \max_{k \in [M]} |y_k| + \max_{j \in [M]} |\Delta_{y_j}| \sum_{i=1}^M |\eta_{i0}^r(z - x - \Delta x) - z_i|.
\]  

(7)

The last line in the above follows from Lemma A.2, for some constant $c > 0$. Now to prove (4) we use the above bound applied to $\text{diff}_{1,k,\ell}$ defined in (3).

\[
|\text{diff}_{1,k,\ell}| \overset{a.s.}{\leq} \frac{c (\log M)^{1/2}}{n^{\delta}} |b_0 \tau_0| \max_{j \in \sec(\ell)} |Z_{0_j}| + \frac{2 \sqrt{n P_\ell}}{\log M} \max_{j \in \sec(\ell)} |\Delta_{10}|_j|,
\]

where we have used the fact that $\sum_{i=1}^M |\eta_{i0}^r(\beta_{0\ell} - a_0 \tau_{00} - a_0 [\Delta_{10}]_\ell) - \beta_0| \leq 2 \sqrt{n P_\ell}$ for each $\ell \in [L]$ and $0 \leq r \leq t$. Now using the above and the results stated in (5) and (6) we find, $\frac{n^\delta}{L} \sum_{\ell=1}^L |\text{diff}_{1,k,\ell}| \overset{a.s.}{\leq} c' n^{\delta - \delta'} \log M + c'' n^{\delta - \delta'}$, which approaches 0 when $\delta < \delta'$.

In what follows we are justified in dropping the deviation terms $\Delta_{10}$. For the second step of the proof, we will appeal to the Strong Law for Triangular Arrays [1] Fact 2 which will show

\[
\lim n^\delta \left[ \frac{1}{L} \sum_{\ell=1}^L \phi_{k,\ell}(a_0 \tau_{00}, b_0 \tau_{00}, \beta_0) - \frac{1}{L} \sum_{\ell=1}^L \mathbb{E}_{Z_0} \{ \phi_{k,\ell}(a_0 \tau_{00}, b_0 \tau_{00}, \beta_0) \} \right] \overset{a.s.}{=} 0.
\]  

(8)

Let $\tilde{Z}_0$ be an independent copy of $Z_0$ and define for each $k = \{1, 2, 3, 4\},$

\[
\text{diff}_{2,k,\ell} := \phi_{k,\ell}(a_0 \tau_{00}, b_0 \tau_{00}, \beta_{\ell0}) - \phi_{k,\ell}(a_0 \tau_{00}, b_0 \tau_{00}, \beta_{00}).
\]  

(9)

In order to use the Strong Law for Triangular Arrays [1] Fact 2 to get result (8) we will prove the following for each function in [1].

\[
\frac{1}{L} \sum_{\ell=1}^L \mathbb{E}_{Z_0} \left| n^\delta \text{diff}_{2,k,\ell} \right|^{2+\kappa} \leq c L^{\kappa/2},
\]  

(10)

for some $0 \leq \kappa \leq 1$ and $c$ some constant. Note that the exact requirement of [1] Fact 2 follows from (10) by an application of Jensen’s Inequality. To prove (10), we consider each $k$ separately. The following result, which follows from (6) for sufficiently large $M$, will be useful. Let $Z \in \mathbb{R}^M$ be a vector of i.i.d. standard Gaussian random variables,

\[
\left( \max_{j \in [M]} |Z_j| \right)^{2+\kappa} = \max_{j \in [M]} |Z_j|^{2+\kappa} \overset{a.s.}{=} \Theta \left( \sqrt{\log M}^{2+\kappa} \right),
\]  

(11)

**k=1.** First we upper bound the difference in (9) as follows:

\[
|\text{diff}_{2,1,\ell}|^{2+\kappa} = |a_0 b_0 \tau_{00}|^{2+\kappa} \left[ \frac{\|Z_0\|_2^2 - \|Z_0\|_2^2}{M^{2+\kappa}} \right] \leq |a_0 b_0 \tau_{00}|^{2+\kappa} \left( \max_{j \in \sec(\ell)} |\tilde{Z}_{0j}|^{2(2+\kappa)} + \max_{j \in \sec(\ell)} |Z_{0j}|^{2(2+\kappa)} \right).
\]  

(12)
From (11), for sufficiently large $M$, each maximum in the above is almost surely $\Theta(\log M^{2+\kappa})$. Therefore, $\frac{1}{L} \sum_{\ell=1}^{L} E_{Z_0,Z_0} |n^\delta \text{diff}_2,k,\ell|^{2+\kappa} \overset{a.s.}{=} \Theta \left( L^\delta \log M^{\delta(2+\kappa)} \right)$. We have satisfied (10) since $M = L^b$ for some constant $b > 0$.

**k=2, 3.** Note that $\phi_{k,\ell}(x, y, \beta_0) \leq \frac{cnP_\ell}{\log M}$ for some constant $c$ when $k = 2, 3$. The considering the difference in (9), we find the following upper bound:

$$
\frac{1}{L} \sum_{\ell=1}^{L} E_{Z_0,Z_0} |n^\delta \text{diff}_2,k,\ell|^{2+\kappa} \leq 2n^\delta (2+\kappa) \left( \frac{cnP_\ell}{\log M} \right)^{2+\kappa} \overset{a.s.}{=} \Theta(n^\delta (2+\kappa)).
$$

Therefore we have satisfied (10).

**k=4.** We first establish an upper bound for $\phi_{4,\ell}(x, y, z)$.

$$
|\phi_{k,\ell}(a_0 \tau_0 Z_0, b_0 \tau_0 Z_0, \beta_0)|^{2+\kappa} = \frac{b_0 \tau_0 Z_0^* \left[ \eta_{k,\ell}^\prime \left( \beta_0 - a_0 \tau_0 Z_0 \right) - \beta_0 \right]}{(\log M)^{2+\kappa}} \overset{a.s.}{=} \Theta \left( \log M^{2+\kappa} \right). \tag{13}
$$

The last equality is true for large enough $M$ by (11). Therefore, $\frac{1}{L} \sum_{\ell=1}^{L} E_{Z_0,Z_0} |n^\delta \text{diff}_2,k,\ell|^{2+\kappa} \overset{a.s.}{=} \Theta(n^\delta (2+\kappa))$, and so we have satisfied (10).

We have satisfied (10) for each function $\phi_{k,\ell}(x, y, z)$ for $k = \{1, 2, 3, 4\}$ and therefore result (8) holds. For the third and final step of the proof, we will provide the following result:

$$
\lim_{\delta} \left[ \frac{1}{L} \sum_{\ell=1}^{L} E_{Z_0} \left\{ \phi_{k,\ell}(a_0 \tau_0 Z_0, b_0 \tau_0 Z_0, \beta_0) \right\} - \frac{1}{L} \sum_{\ell=1}^{L} E_{(Z_0,\beta)} \left\{ \phi_{k,\ell}(a_0 \tau_0 Z_0, b_0 \tau_0 Z_0, \beta_0) \right\} \right] \overset{a.s.}{=} 0.
$$

For each function $\phi_{k,\ell}(x, y, z)$ with $k = \{1, 2, 3, 4\}$ and each $\beta_0 \in B_{M,L}$, we prove the following for each $\ell \in [L]$:

$$
E_{Z_0} \left\{ \phi_{k,\ell}(a_0 \tau_0 Z_0, b_0 \tau_0 Z_0, \beta_0) \right\} = E_{(Z_0,\beta)} \left\{ \phi_{k,\ell}(a_0 \tau_0 Z_0, b_0 \tau_0 Z_0, \beta_0) \right\}.
$$

The key is that the random variables $\{Z_0\}$ are i.i.d. across $j \in \sec(\ell)$, and the position of the non-zero entry in $\beta_\ell$ is uniformly distributed across $j \in [M]$.

**k=1.** The result is trivially true since $\phi_{k,\ell}(x, y, z)$ doesn’t depend on $z$.

**k=2, 3.** First consider the $k = 2$ function

$$
\phi_{2,\ell}(a_0 \tau_0 Z_0, b_0 \tau_0 Z_0, \beta_0) = \frac{\|\eta_0^\beta - a_0 \tau_0 Z_0\|^2}{\log M}.
$$

Let $k \in \sec(\ell)$ be the non-zero element of $\beta_0$. Then for $i \in \sec(\ell)$

$$
\eta_0^\beta - a_0 \tau_0 Z_0 = \sqrt{n P_\ell} \exp \left( \frac{n P_\ell}{\tau_0} \cdot \mathbb{I}(i = k) \right) \exp \left( -\sqrt{n P_\ell} a_0 Z_0 \right) \exp \left( -\sqrt{n P_\ell} a_0 Z_0 \right) + \sum_{j \neq k} \exp \left( -\sqrt{n P_\ell} a_0 Z_0 \right).
$$
Therefore,
\[ \mathbb{E}_{Z_0} \{ \phi_{k,\ell} (a_0, b_0, \beta_0) \} = \frac{n \mathcal{P}_t}{\log M} \mathbb{E}_{Z_0} \left[ \exp \left( \frac{2n \mathcal{P}_t}{\gamma_0} \right) \exp \left( -2\sqrt{\frac{n \mathcal{P}_t}{\gamma_0}} a_0 Z_{0k} \right) + \sum_{i \neq k} \exp \left( -2\sqrt{\frac{n \mathcal{P}_t}{\gamma_0}} a_0 Z_{0k} \right) \right] \right] ^{\frac{1}{2}} \right]. \tag{14}

The key observation is that the expectation on the RHS of (14) is the same regardless of whether the non-zero index \( i \) in \( \beta_0 \) is 1, 2, \ldots, or \( M \). This is because \( \{Z_0\} \) is i.i.d. across \( j \in \sec(\ell) \). Hence,
\[ \mathbb{E}_{(Z_0, \beta)} \{ \phi_{k,\ell} (a_0, b_0, \beta) \} = \sum_{i=1}^{M} \frac{1}{M} \mathbb{E}_{Z_0} \{ \phi_{k,\ell} (a_0, b_0, \beta) \} \text{Non-zero entry of } \beta \text{ is } i \}.
\]
The above equals (14). The argument for the \( k = 3 \) function \( \phi_{3,\ell}(x, y, z) \) similar.

**k=4.** The result can be shown in a manner similar to that used for the \( k = 2 \) function \( \phi_{2,\ell}(x, y, z) \) shown above.

The existence of the limit of \( \frac{1}{t} \sum_{\ell=1}^{L} \mathbb{E}_{(Z_0, \beta)} \{ \phi_{k,\ell}(a_0, b_0, \beta) \} \) for \( k = 1 \) follows from the law of large numbers; for \( k = 2, 3, 4 \), the limit is derived in Appendix A.4 in [1].

\( \square \)

2 Step 4: Showing \( H_{t+1}(b)(i) \) holds

We want to show that if [1] \( B_{r} \) and [1] \( H_{s} \) hold for \( 0 \leq r < t \leq T^* \) and \( 1 \leq s < t \leq T^* \) then [2] holds.

**Proof.** Consider the functions \( \phi_{k,\ell}(x, y, z) \) for \( k \in \{1,2,3,4\} \) defined in [1]. First note that the result of this Lemma, given in [2], is true for an additional group of functions defined as follows:
\[ \phi_{5,\ell}(x, y, z) = \mathbb{E}_Z \{ \phi_{k,\ell}(x + \sigma Z, y + \sigma Z, z) \}, \]
where \( k = \{1,2,3,4\} \), random vector \( Z \in \mathbb{R}^M \) is i.i.d. \( \sim \mathcal{N}(0,1) \) independent of \( x, y, z \), and \( \lim \sigma Z \) is constant. We do not prove the result explicitly for these functions since it follows by application of Jensen’s inequality. In what follows we prove the result for generic \( \phi_{k,\ell}(x, y, z) \) with \( k \in \{1,2,3,4\} \) and we state explicitly which \( k \) we refer to when it is important to the results.

From [1] Lemma 4 it follows,
\[ \phi_{k,\ell} \left( \sum_{u=0}^{t} a_u h_{\ell}^{u+1}, \sum_{v=0}^{t} b_v h_{\ell}^{v+1}, \beta_0 \right) \mid_{\mathcal{A}_{t+1},t} \]
\[ \overset{d}{=} \phi_{k,\ell} \left( \sum_{u=0}^{t-1} a_u h_{\ell}^{u+1} + a_t \bar{\tau}_{\ell}^+ Z_{t\ell} + a_t[\Delta_{t+1},\ell]t, \sum_{v=0}^{t-1} b_v h_{\ell}^{v+1} + b_t \bar{\tau}_{\ell}^+ Z_{t\ell} + b_t[\Delta_{t+1},\ell]t, \beta_0 \right), \]
where \( a_u' = a_u \) and \( b_v' = b_v \) for \( 0 \leq u, v \leq t - 2 \) and \( a_u' = a_{t-1} + a_t(\bar{\tau}^2_{\ell}/\bar{\tau}^2_{t-1}) \) and \( b_v' = b_{t-1} + b_t(\bar{\tau}^2_{\ell}/\bar{\tau}^2_{t-1}) \). Define
\[ \text{diff}_{1,k,\ell} := \phi_{k,\ell} \left( \sum_{u=0}^{t-1} a'_u h_{\ell}^{u+1} + a_t \bar{\tau}_{\ell}^+ Z_{t\ell} + a_t[\Delta_{t+1},\ell]t, \sum_{v=0}^{t-1} b'_v h_{\ell}^{v+1} + b_t \bar{\tau}_{\ell}^+ Z_{t\ell} + b_t[\Delta_{t+1},\ell]t, \beta_0 \right) - \phi_{k,\ell} \left( \sum_{u=0}^{t-1} a'_u h_{\ell}^{u+1} + a_t \bar{\tau}_{\ell}^+ Z_{t\ell}, \sum_{v=0}^{t-1} b'_v h_{\ell}^{v+1} + b_t \bar{\tau}_{\ell}^+ Z_{t\ell}, \beta_0 \right), \tag{15} \]
First, we show that the deviation term $\Delta_{t+1,t}$ can be dropped when considering the limit. Considering $t$ fixed, for each function in $[\mathcal{H}]$ we first show the following.

$$\lim n^\delta \frac{1}{L} \sum_{\ell=1}^L |\text{diff}_{1,k,\ell}|^\alpha_{n,\ell} = 0. \quad (16)$$

for some $\delta > 0$. To prove the above we will supply upper bounds on the difference $\text{diff}_{1,k,\ell}$ defined in $(15)$ which approach 0 almost surely in the limit. To do so, we consider each $k$ separately. The following two results are useful:

1. From $[\mathcal{H}]$ $\mathcal{H}_1(a) - \mathcal{H}_{t+1}(a)$, for each $\ell \in [L]$ and for constant $c_1 > 0$ not depending on $N$,

$$\max_{j \in \sec(\ell)} |\Delta_{t+1,t,j}| \frac{\alpha_{n,\ell}}{n^{\delta'}} = \Theta(n^{-\delta'} \sqrt{\log M}), \quad \text{and} \quad \max_{j' \in \sec(\ell)} |h_{j'}^{r+1}| \leq c_1 \sqrt{\log M} \quad 0 \leq r \leq t. \quad (17)$$

2. Result $(17)$ along with Lemma $6$ implies, for each $\ell \in [L]$ and for some constant $C > 0$, that

$$\max_{j \in \sec(\ell)} \left| \sum_{u=0}^{t-1} a'_u h_j^{u+1} + a_t \bar{\tau}_j Z_{t,j} \right| \leq C \sqrt{\log M} \quad \text{for all sufficiently large } M. \quad (18)$$

$k = 1$. Because of the $\Delta_{t+1,t}$ bounds given in $(17)$, it follows from Lemma $(A.1)$ for each $\ell \in [L]$,

$$|\text{diff}_{1,k,\ell}| \leq \frac{C \sqrt{\log M}}{n^{\delta'}} \left[ \max_{j \in \sec(\ell)} \left| \sum_{u=0}^{t-1} a'_u h_j^{u+1} + a_t \bar{\tau}_j Z_{t,j} \right| + \max_{j' \in \sec(\ell)} \left| \sum_{v=0}^{t-1} b'_v h_j^{v+1} + b_t \bar{\tau}_j Z_{t,j} \right| + \frac{\sqrt{\log M}}{n^{\delta'}} \right],$$

Using bound $(18)$ in the above, we find $\frac{n^\delta}{L} \sum_{\ell=1}^L |\text{diff}_{1,k,\ell}| \leq C' n^{\delta - \delta'} \log M$, which approaches 0 when $\delta < \delta'$ giving result $(16)$.

$k = 2,3$. Because of the $\Delta_{t+1,t}$ bounds given in $(17)$, it follows from Lemma $(A.2)$ for each $\ell \in [L]$, $|\text{diff}_{1,k,\ell}| \leq C n^{-\delta} \log M$. Plugging this into $(16)$,

$$\frac{n^\delta}{L} \sum_{\ell=1}^L |\text{diff}_{1,k,\ell}| \leq C' n^{\delta - \delta'} \log M.$$

The right-side of the above approaches 0 when $\delta < \delta'$.

$k = 4$. Using the upper bound for $\phi_{\delta,\ell}(x,y,z)$ provided in $(7)$, for constants $c,C > 0$,

$$(\log M) |\text{diff}_{1,k,\ell}| \leq C \sqrt{\log M} \cdot \frac{c(\log M)^{3/2}}{n^{\delta'}} + 2 \sqrt{n P} \cdot \Theta(n^{-\delta'} \sqrt{\log M}),$$

where we have used results $(17)$, $(18)$, and Lemma $(A.2)$. Now using the above in $(16)$, we find

$$\frac{n^\delta}{L} \sum_{\ell=1}^L |\text{diff}_{1,k,\ell}| \leq C' n^{\delta - \delta'} \log M + c'' n^{\delta - \delta'},$$

which approaches 0 when $\delta < \delta'$.

In what we are justified in dropping the deviation terms $\Delta_{t+1,t}$. For the second step of the proof, we will appeal to the Strong Law for Triangular Arrays $[\mathcal{H}]$ Fact 2 which will show

$$\lim n^\delta \left[ \frac{1}{L} \sum_{\ell=1}^L \phi_{k,\ell} \left( \sum_{u=0}^{t-1} a'_u h_j^{u+1} + a_t \bar{\tau}_j Z_{t,j}, \sum_{v=0}^{t-1} b'_v h_j^{v+1} + b_t \bar{\tau}_j Z_{t,j}, \beta_{0,t} \right) \right] = 0. \quad (19)$$
Let \( \tilde{Z}_t \) be an independent copy of \( Z_t \) and define for each \( k = \{1, 2, 3, 4\} \),

\[
\text{diff}_{2,k,t} := \phi_{k,t} \left( \sum_{u=0}^{t-1} a'_u h^{u+1}_t + a_t \bar{\tau}_t^\perp \tilde{Z}_t, \sum_{v=0}^{t-1} b'_v h^{v+1}_t + b_t \bar{\tau}_t^\perp \tilde{Z}_t, \beta_{0t} \right)
- \phi_{k,t} \left( \sum_{u=0}^{t-1} a'_u h^{u+1}_t + a_t \bar{\tau}_t^\perp Z_t, \sum_{v=0}^{t-1} b'_v h^{v+1}_t + b_t \bar{\tau}_t^\perp Z_t, \beta_{0t} \right). \tag{20}
\]

In order to use the Strong Law for Triangular Arrays \([1, \text{Fact 2}]\) to get result \([19]\), we will prove the following for each function in \([1]\).

\[
\frac{1}{L} \sum_{\ell=1}^{L} \mathbb{E}_{\tilde{Z}_t, Z_t} \left[ n^2 \text{diff}_{2,k,t} \right]^{2+\kappa} \leq cL^{\kappa/2}, \tag{21}
\]

for some \( 0 \leq \kappa \leq 1 \) and \( c \) some constant. Note that the exact requirement of \([1, \text{Fact 2}]\) is met by an application of Jensen’s Inequality. To prove \((21)\), we consider each \( k \) separately. The following result, which follows from \([4]\) for sufficiently large \( M \), will be useful. Let \( Z \in \mathbb{R}^M \) be a vector of i.i.d. standard Gaussian random variables,

\[
\left( \max_{j \in \sec(t)} \left| \sum_{u=0}^{t-1} a'_u h^{u+1}_j + a_t \bar{\tau}_t^\perp Z_j \right| \right)^{2+\kappa} = \max_{j' \in \sec(t)} \left| \sum_{u=0}^{t-1} a'_u h^{u+1}_{j'} + a_t \bar{\tau}_t^\perp \tilde{Z}_t \right| \stackrel{a.s.}{=} \Theta \left( \sqrt{\log M}^{2+\kappa} \right). \tag{22}
\]

\( k = 1. \) As in \([12]\), first we upper bound the difference in \((20)\) as follows:

\[
|\text{diff}_{2,k,t}|^{2+\kappa} \leq \max_{j \in \sec(t)} \left| \sum_{u=0}^{t-1} a'_u h^{u+1}_j + a_t \bar{\tau}_t^\perp \tilde{Z}_t \right|^{2+\kappa} \max_{j' \in \sec(t)} \left| \sum_{v=0}^{t-1} b'_v h^{v+1}_{j'} + b_t \bar{\tau}_t^\perp \tilde{Z}_t \right|^{2+\kappa}
+ \max_{j \in \sec(t)} \left| \sum_{u=0}^{t-1} a'_u h^{u+1}_j + a_t \bar{\tau}_t^\perp Z_t \right|^{2+\kappa} \max_{j' \in \sec(t)} \left| \sum_{v=0}^{t-1} b'_v h^{v+1}_{j'} + b_t \bar{\tau}_t^\perp Z_t \right|^{2+\kappa}.
\]

From \((22)\), for sufficiently large \( M \), each maximum in the above is almost surely \( \Theta \left( \sqrt{\log M}^{2+\kappa} \right) \). Therefore, \( \frac{1}{L} \sum_{\ell=1}^{L} \mathbb{E}_{Z_0, Z_0} \left[ n^2 \text{diff}_{2,k,t} \right]^{2+\kappa} \stackrel{a.s.}{=} \Theta \left( L^\delta (\log M)^{\delta(2+\kappa)} \right) \). We have satisfied \((21)\) since \( M = L^b \) for some constant \( b > 0 \).

\( k = 2, 3. \) Note that \( \phi_{k,t}(x, y, \beta_{0t}) \leq \frac{cnP_k}{\log M} \) for some constant \( c \) when \( k = 2, 3 \). The considering the difference in \((20)\), we find the following upper bound:

\[
\frac{1}{L} \sum_{\ell=1}^{L} \mathbb{E}_{\tilde{Z}_t, Z_t} \left[ n^2 \text{diff}_{2,k,t} \right]^{2+\kappa} \leq 2 n^{\delta(2+\kappa)} \left( \frac{cnP_k}{\log M} \right)^{2+\kappa} \stackrel{a.s.}{=} \Theta(n^{\delta(2+\kappa)}).
\]

Therefore we have satisfied \((21)\).
\( k=4 \). As in (13) we first establish an upper bound for \( \phi_{4,\ell}(x, y, z) \),

\[
\left| \phi_{k,\ell} \left( \sum_{u=0}^{t-1} a'_u h_{\ell}^{v+1} + a_t \tau_t^Z, \sum_{v=0}^{t-1} b'_v h_{\ell}^{v+1} + b_t \tau_t^Z, \beta_{0,\ell} \right) \right|^{2+\kappa} \leq \left( \frac{2\sqrt{nP_{\ell}}}{(\log M)^{2+\kappa}} \right)^{2+\kappa} \max_{j \in \text{sec}(\ell)} \left| \sum_{v=0}^{t-1} b'_v h_{\ell}^{v+1} + b_t \tau_t^Z \right|^{2+\kappa} \leq \frac{1}{\ell} \sum_{\ell=1}^{L} \mathbb{E}_{Z_t, Z_{\ell}} \left| n^{\delta} \text{diff}_{2, k, \ell} \right|^{2+\kappa} \leq 2n^{\delta(2+\kappa)} \left( \frac{cn_{\ell}}{\log M} \right)^{2+\kappa} a.s. \Theta(n^{\delta(2+\kappa)}). \]

The last equality is true, for sufficiently large \( M \), by (22). Therefore, \( \frac{1}{\ell} \sum_{\ell=1}^{L} \mathbb{E}_{Z_t, Z_{\ell}} \left| n^{\delta} \text{diff}_{2, k, \ell} \right|^{2+\kappa} \leq 2n^{\delta(2+\kappa)} \left( \frac{cn_{\ell}}{\log M} \right)^{2+\kappa} a.s. \). We have satisfied (21) for Class 2 functions.

We have satisfied (21) for each function \( \phi_{k,\ell}(x, y, z) \) for \( k = \{1, 2, 3, 4\} \) and therefore result (19) holds. Considering result (19), define new functions \( \phi_{k,\ell}^{NEW} \) for \( k = \{1, 2, 3, 4\} \) as

\[
\phi_{k,\ell}^{NEW} \left( \sum_{u=0}^{t-1} a'_u h_{\ell}^{v+1}, \sum_{v=0}^{t-1} b'_v h_{\ell}^{v+1}, \beta_{0,\ell} \right) := \mathbb{E}_{Z_t} \left\{ \phi_{k,\ell} \left( \sum_{u=0}^{t-1} a'_u h_{\ell}^{v+1} + a_t \tau_t^Z, \sum_{v=0}^{t-1} b'_v h_{\ell}^{v+1} + b_t \tau_t^Z, \beta_{0,\ell} \right) \right\}.
\]

Using Jensen’s inequality, it can be shown that the induction hypothesis \([1, \mathcal{H}_t(b)]\) holds for the function \( \phi_{k,\ell}^{NEW} \) whenever \( \mathcal{H}_t(b) \) holds for the function \( \phi_{k,\ell} \) inside the expectation. Therefore,

\[
\lim n^{\delta} \left[ \frac{1}{L} \sum_{\ell=1}^{L} \mathbb{E}_{Z_t} \left\{ \phi_{k,\ell} \left( \sum_{u=0}^{t-1} a'_u h_{\ell}^{v+1} + a_t \tau_t^Z, \sum_{v=0}^{t-1} b'_v h_{\ell}^{v+1} + b_t \tau_t^Z, \beta_{0,\ell} \right) \right\} - \frac{1}{L} \sum_{\ell=1}^{L} \mathbb{E}_{Z_t} \left\{ \phi_{k,\ell} \left( \sum_{u=0}^{t-1} a'_u \tau_u Z_u + a_t \tau_t^Z, \sum_{v=0}^{t-1} b'_v \tau_v Z_v + b_t \tau_t^Z, \beta_{0,\ell} \right) \right\} \right] a.s. \equiv 0.
\]

To complete the proof we show that

\[
\mathbb{E}_{Z_t} \left\{ \phi_{k,\ell} \left( \sum_{u=0}^{t-1} a'_u \tilde{Z}_u + a_t \tilde{Z}_t, \sum_{v=0}^{t-1} b'_v \tilde{Z}_v + b_t \tilde{Z}_t, \beta_{\ell} \right) \right\} = \mathbb{E} \left\{ \phi_{k,\ell} \left( \sum_{u=0}^{t} a'_u \tilde{Z}_u, \sum_{v=0}^{t} b'_v \tilde{Z}_v, \beta_{\ell} \right) \right\}.
\]

Recall \( a'_{t-1} = a'_{t-1} = a_t + a_t (\tilde{\tau}_r^2 / \tilde{\tau}_r^1) \) and \( b'_{t-1} = b_t + b_t (\tilde{\tau}_r^2 / \tilde{\tau}_r^1) \). Then to prove the above we will show that \((\tilde{\tau}_r^2 / \tilde{\tau}_r^1) \tilde{Z}_{t-1} + \tilde{\tau}_t^1 \tilde{Z}_t \overset{d}{=} \tilde{\tau} \tilde{Z}_t \) where \( \tilde{\tau} \tilde{\tau}_r \mathbb{E} [ \tilde{Z}_r \tilde{Z}_t ] = \tilde{\tau}_r^2 \) for \( 0 \leq r \leq t-1 \). Note that \((\tilde{\tau}_r^2 / \tilde{\tau}_r^1) \tilde{Z}_{t-1} + \tilde{\tau}_t^1 \tilde{Z}_t \) is Gaussian with variance equal to \((\tilde{\tau}_r^2 / \tilde{\tau}_r^1)^2 + (\tilde{\tau}_t^1)^2 = \tilde{\tau}_r^2 \) using the definition of \( \tilde{\tau}_t^1 \). This follows since \( \tilde{Z}_{t-1} \) and \( Z_t \) are independent. Finally, for \( 0 \leq r \leq t-1 \)

\[
\mathbb{E} \left\{ \tilde{\tau}_r \tilde{Z}_r \left( (\tilde{\tau}_r^2 / \tilde{\tau}_t^1) \tilde{Z}_{t-1} + \tilde{\tau}_t^1 \tilde{Z}_t \right) \right\} = (\tilde{\tau}_r^2 / \tilde{\tau}_t^1) \tilde{\tau}_r \tilde{\tau}_{t-1} \mathbb{E} [ \tilde{Z}_r \tilde{Z}_{t-1} ] = \tilde{\tau}_r^2.
\]

The existence of the limit of \( \mathbb{E} \{ \phi_{k,\ell} (\sum_{u=0}^{t} a'_u \tilde{Z}_u, \sum_{v=0}^{t} b'_v \tilde{Z}_v, \beta_{\ell} ) \} \) for \( k = 1 \) follows from the law of large numbers; for \( k = 2, 3, 4 \), the existence of the limit follows from Appendix A.4 in [1]. This completes the proof.

\[ \blacksquare \]
A Appendix

Lemma A.1 ($k = 1$ Function Bound). We consider the function $\phi_{1, \ell} : \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}$ defined as $\phi_{1, \ell}(x, y, z) = x^*y/M$. If $\max_{j \in [M]} |\Delta_{x_j}|$ and $\max_{j \in [M]} |\Delta_{y_j}|$ are both almost surely $\Theta(n^{-\delta}\sqrt{\log M})$ for some $\delta > 0$, the following holds for some constant $C > 0$:

$$\left| \phi_h(x, y, z) - \phi_h(x + \Delta_x, y + \Delta_y, z) \right| \leq \frac{C\sqrt{\log M}}{n^\delta} \left[ \max_{j \in [M]} |x_j| + \max_{j' \in [M]} |y_{j'}| + \frac{\sqrt{\log M}}{n^\delta} \right].$$

(23)

Proof.

$$\left| \phi_h(x, y, z) - \phi_h(x + \Delta_x, y + \Delta_y, z) \right| = \frac{1}{M} |x^*y - (x + \Delta_x)^*(y + \Delta_y)|$$

$$\leq \frac{1}{M} |x^*\Delta_y| + \frac{1}{M} |\Delta_x^*(y + \Delta_y)|$$

$$\leq \max_{j \in [M]} |\Delta_{y_j}| \max_{i \in [M]} |x_i| + \max_{j \in [M]} |\Delta_{x_j}| \left( \max_{i \in [M]} |y_i| + \max_{j' \in [M]} |\Delta_{y_{j'}}| \right).$$

The result follows since $\max_{j \in [M]} |\Delta_{x_j}|$ and $\max_{j \in [M]} |\Delta_{y_j}|$ are both almost surely $\Theta(n^{-\delta}\sqrt{\log M})$.

\[ \square \]

Lemma A.2 ($k = 2, 3$ Function Bound). We consider the functions $\phi_{k, \ell} : \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}$ for $k = 2, 3$ defined as

$$\phi_{2, \ell}(x, y, z) := \|\eta^\ell_1(z - x)\|^2 / \log M,$$

$$\phi_{3, \ell}(x, y, z) := \|\eta^\ell_1(z - x) - z\| \log M,$$

(24)

If $\max_{j \in [M]} |z^j_1| = \sqrt{nP_\ell}$ and $\max_{j \in [M]} |\Delta_{x_j}|$ and $\max_{j \in [M]} |\Delta_{y_j}|$ are both almost surely $\Theta(n^{-\delta}\sqrt{\log M})$ for some $\delta > 0$ and each $\ell \in [L]$, the following holds for some constant $C > 0$, for $k = 2, 3$:

$$\left| \phi_{k, \ell}(x, y, z) - \phi_{k, \ell}(x + \Delta_x, y + \Delta_y, z) \right| \leq Cn^{-\delta} \log M.$$

Furthermore for $0 \leq u \leq t$,

$$\sum_{i=1}^M |\eta^u_i(z - x - \Delta_x) - \eta^u_i(z - x)| \leq Cn^{-\delta}(\log M)^{3/2}. \quad (25)$$

Proof. In the following assume $\ell \in [L]$ is fixed, and therefore for $i \in [M]$ (and $i \in \text{sec}(\ell)$) we let,

$$\eta^u_1(v) = \sqrt{nP_\ell} \exp \left\{ \frac{v_1\sqrt{nP_\ell}}{\tau_2^2} \right\} \left[ \sum_{j=1}^M \exp \left\{ \frac{v_j\sqrt{nP_\ell}}{\tau_2^2} \right\} \right]^{-1}. \quad (26)$$

Define the following:

$$F_{c, \Delta, \tau_2}^1 := \exp \left\{ \frac{c\sqrt{nP_\ell}}{\tau_2^2} \max_{j \in [M]} |\Delta_j| \right\} - 1, \quad \text{and} \quad F_{c, \Delta, \tau_2}^2 := 1 - \exp \left\{ -\frac{c\sqrt{nP_\ell}}{\tau_2^2} \max_{j \in [M]} |\Delta_j| \right\}. \quad (26)$$

In what follows we upper bound both functions $k = 2, 3$ using the above definition as follows for some constant $C, c > 0$:

$$\left| \phi_{k, \ell}(x, y, z) - \phi_{k, \ell}(x + \Delta_x, y + \Delta_y, z) \right| \leq \frac{Cn^\ell}{\sqrt{\log M}} \left( \max \left( F_{c, \Delta, \tau_2}^1, F_{c, \Delta, \tau_2}^1, F_{c, \Delta, \tau_2}^2, F_{c, \Delta, \tau_2}^2, F_{c, \Delta, \tau_2}^2, F_{c, \Delta, \tau_2}^2 \right). \quad (27)$$
Then a Taylor expansion of $e^x$ can be used to show that each of $F_{c,\Delta x, \tau^2}^1, F_{c,\Delta y, \tau^2}^1, F_{c,\Delta x, \tau^2}^2, F_{c,\Delta y, \tau^2}^2$ can be upper bounded almost surely by $c'n^{-\delta} \log M$ whenever $\max_{j \in [M]} |\Delta x_j|$ and $\max_{j \in [M]} |\Delta y_j|$ are both almost surely $\Theta(n^{-\delta} \sqrt{\log M})$. Therefore

$$\max(F_{c,\Delta x, \tau^2}^1, F_{c,\Delta y, \tau^2}^1, F_{c,\Delta x, \tau^2}^2, F_{c,\Delta y, \tau^2}^2) \leq C'n^{-\delta} \log M,$$

for constant $C' > 0$, which along with the bound in (27) provides the desired result. What remains is to prove (27) for $k = 2, 3$.

Now to complete the proof we show that upper bound (27) for both functions $k = 2, 3$.

**k=2** First note,

$$\phi_{2,\ell}(x, y, z) = \frac{||\eta'(z - x)||^2}{\log M} = \frac{n P_\ell}{\log M} \sum_{i=1}^{M} \exp \left\{ \frac{2(z_i - x_i) \sqrt{n P_\ell}}{\tau^2} \right\} \left( \sum_{j=1}^{M} \exp \left\{ \frac{2\Delta x_i \sqrt{n P_\ell}}{\tau^2} \right\} \right)^2.$$

From the above we can write

$$\log M \frac{\phi_{2,\ell}(x, y, z)}{n P_\ell} = \sum_{i=1}^{M} \exp \left\{ \frac{2(z_i - x_i - \Delta x_i) \sqrt{n P_\ell}}{\tau^2} \right\} \left( \sum_{j=1}^{M} \exp \left\{ \frac{2\Delta x_i \sqrt{n P_\ell}}{\tau^2} \right\} \right)^2 \leq \sum_{i=1}^{M} \exp \left\{ \frac{2(z_i - x_i - \Delta x_i) \sqrt{n P_\ell}}{\tau^2} \right\} \left( \sum_{j=1}^{M} \exp \left\{ \frac{2\max_{j' \in [M]} |\Delta x_{j'}| \sqrt{n P_\ell}}{\tau^2} \right\} \right)^2.$$

and similarly

$$\log M \frac{\phi_{2,\ell}(x, y, z)}{n P_\ell} \geq \sum_{i=1}^{M} \exp \left\{ \frac{2(z_i - x_i - \Delta x_i) \sqrt{n P_\ell}}{\tau^2} \right\} \left( \sum_{j=1}^{M} \exp \left\{ \frac{2\max_{j' \in [M]} |\Delta x_{j'}| \sqrt{n P_\ell}}{\tau^2} \right\} \right)^2.$$

Putting (29) and (30) together we see that

$$\exp \left\{ -\frac{4\sqrt{n P_\ell}}{\tau^2} \max_{j' \in [M]} |\Delta x_{j'}| \right\} \leq \frac{\phi_{2,\ell}(x, y, z)}{\phi_{2,\ell}(x + \Delta x, y + \Delta y, z)} \leq \exp \left\{ \frac{4\sqrt{n P_\ell}}{\tau^2} \max_{j' \in [M]} |\Delta x_{j'}| \right\},$$

and so we find the desired result (27) for $k=2$:

$$|\phi_{2,\ell}(x, y, z) - \phi_{2,\ell}(x + \Delta x, y + \Delta y, z)| \leq \phi_{2,\ell}(x + \Delta x, y + \Delta y, z) \max \left( F_{4,\Delta x, \tau^2}^1, F_{4,\Delta y, \tau^2}^2 \right) \leq \frac{n P_\ell}{\log M} \max \left( F_{4,\Delta x, \tau^2}^1, F_{4,\Delta y, \tau^2}^2 \right) .$$

**k=3** First note,

$$\log M \left[ \phi_{3,\ell}(x, y, z) - \phi_{3,\ell}(x + \Delta x, y + \Delta y, z) \right] = \eta'(z - x)^* \eta^2(z - y) - \eta^2(z - x - \Delta x)^* \eta^2(z - y - \Delta y) - z^* \left[ \eta^2(z - x) - \eta^2(z - x - \Delta x) \right] z^* \left[ \eta^2(z - y) - \eta^2(z - y - \Delta y) \right] .$$

We again suppress the explicit notation for the dependence on section $\ell$. Then to prove result (27) we prove the following two upper bounds.
1. For $0 \leq r \leq t$,

$$|z^* [\eta^r(z - x) - \eta^r(z - x - \Delta_x)]| \leq nP_t \max \left(F_{2,\Delta_x,\tau_x^2}^{1}, F_{2,\Delta_x,\tau_x^2}^{2}\right). \quad (34)$$

2. For $0 \leq r \leq s \leq t$,

$$|\eta^r(z - x)^s \eta^s(z - y) - \eta^r(z - x - \Delta_x)^s \eta^s(z - y - \Delta_y)| \leq 2nP_t \max \left(F_{2,\Delta_x,\tau_x^2}^{1}, F_{2,\Delta_y,\tau_y^2}^{1}, F_{2,\Delta_x,\tau_x^2}^{2}, F_{2,\Delta_y,\tau_y^2}^{2}\right). \quad (35)$$

Both bounds are trivial for the $r = -1$ case. Using (34) and (35) in (33) we get the desired result (27). First consider (34):

$$|z^* [\eta^r(z - x - \Delta_x) - \eta^r(z - x)]| \leq \max_{j \in [M]} z_j \sum_{i=1}^{M} |\eta^r_i(z - x - \Delta_x) - \eta^r_i(z - x)|$$

$$= \sqrt{nP_t} \sum_{i=1}^{M} |\eta^r_i(z - x - \Delta_x) - \eta^r_i(z - x)|. \quad (36)$$

Next consider (35):

$$|\eta^r(z - x)^s \eta^s(z - y) - \eta^r(z - x - \Delta_x)^s \eta^s(z - y - \Delta_y)| \leq |\eta^r(z - x)^s \eta^s(z - y) - \eta^r(z - y - \Delta_y)| + |\eta^r(z - x - \Delta_x)^s \eta^s(z - y - \Delta_y)|$$

$$\leq \sqrt{nP_t} \sum_{i=1}^{M} |\eta^s_i(z - y) - \eta^s_i(z - y - \Delta_y)| + \sqrt{nP_t} \sum_{i=1}^{M} |\eta^r_i(z - x) - \eta^r_i(z - x - \Delta_x)|. \quad (37)$$

Now considering (36) and (37), to get the results given in (34) and (35) we demonstrate the following for $0 \leq u \leq t$,

$$\sum_{i=1}^{M} |\eta^u_i(z - x - \Delta_x) - \eta^u_i(z - x)| \leq \sqrt{nP_t} \max \left(F_{2,\Delta_x,\tau_x^2}^{1}, F_{2,\Delta_x,\tau_x^2}^{2}\right). \quad (38)$$

Note that the above leads to result (25) using (28). We now prove (38). Using a strategy similar to that in in (31),

$$\exp \left\{-\frac{2\sqrt{nP_t}}{\tau_x^2} \max_{j \in [M]} |\Delta_{x_j}'|\right\} \leq \frac{\eta^u_i(z - x)}{\eta^u_i(z - x - \Delta_x)} \leq \exp \left\{\frac{2\sqrt{nP_t}}{\tau_x^2} \max_{j \in [M]} |\Delta_{x_j}'|\right\}.$$ 

Result (38) follows from the above as in (32).

References