DESIGN OF Q-SHIFT COMPLEX WAVELETS FOR IMAGE PROCESSING USING FREQUENCY DOMAIN ENERGY MINIMIZATION

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ABSTRACT

This paper proposes a new method of designing finite-support wavelet filters, based on minimization of energy in key parts of the frequency domain. In particular this technique is shown to be very effective for designing families of filters that are suitable for use in the shift-invariant Dual-Tree Complex Wavelet structure that has been developed by the author recently, and has been shown to be important for a range of image processing applications. The Dual-Tree structure requires most of the wavelet filters to have a well-controlled group delay, equivalent to one quarter of a sample period, in order to achieve optimal shift invariance. The proposed new design technique allows this requirement to be included along with the usual smoothness and perfect reconstruction properties to yield wavelet filters with a unique combination of features: linear phase, tight frame, compact spatial support, good frequency domain selectivity with low sidelobe levels, approximate shift invariance, and good directional selectivity in two or more dimensions.

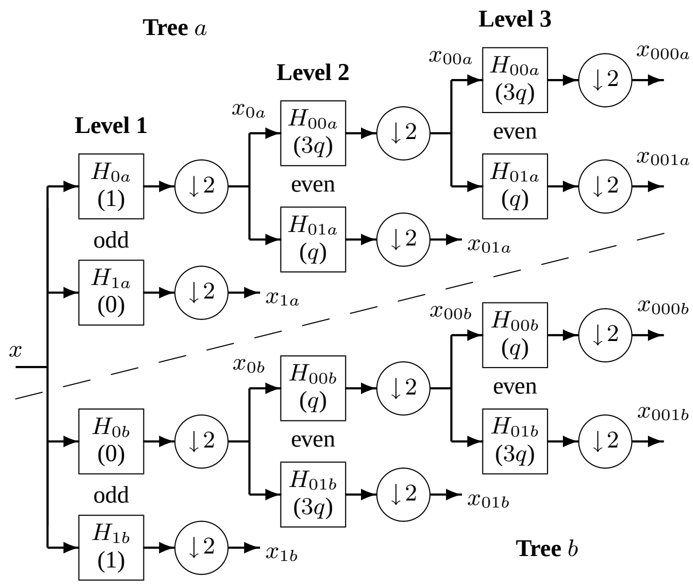

Fig. 1. Dual-Tree Complex Wavelet Transform over 3 levels.

1. INTRODUCTION

The Dual-Tree Complex Wavelet Transform (DT CWT), as shown in fig. 1, was introduced in [1] in order to provide the following significant improvements over the usual fully-decimated discrete wavelet transform (DWT): approximate shift invariance, and good directional selectivity for images and other multidimensional signals. The penalty paid for this is a 2:1 increase in redundancy per signal dimension (ie 4:1 for 2D signals, 8:1 for 3D etc), but this is much less redundant than the undecimated (or continuous) DWT. The redundancy is introduced through the addition of a second tree of subband filters (tree b), shown below the dashed line in fig. 1.

In [1] we showed that the key to obtaining shift invariance from the dual tree structure lies in designing the filter delays at each stage, such that the lowpass filter outputs in tree b are sampled at points midway between the sampling points of the equivalent filters in tree a. This requires a delay difference between the a and b lowpass filters of 1 sample period at tree level 1, and of $\frac{1}{2}$ sample period at subsequent levels. At level 1 we can use any standard orthogonal or bi-orthogonal wavelet filters, and produce the required delay shift trivially by insertion or deletion of unit delays, but at further levels the $\frac{1}{2}$ sample delay difference is more difficult. In [2], Selesnick showed that these lowpass delay constraints produce a Hilbert pair relationship between the wavelet bases for the two trees. This leads naturally to the interpretation of the outputs from trees a and b as the real and imaginary parts respectively of *complex* wavelet coefficients.

In [3, 4] we introduced Q-shift filters for levels 2 and below, in order to give the dual-tree improved orthogonality and symmetry properties over an earlier form. This paper describes a new and effective design technique for optimizing the Q-shift filters, particularly for image processing applications where linear-phase is an important property of the resulting complex wavelets and scaling functions. Our previous published design approach, proposed in [3, 4], was based on optimization of a set of rotations θ_i in a polyphase matrix factorization of the filters, but this is

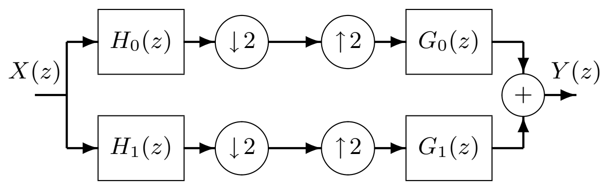

Fig. 2. 2-band analysis and reconstruction filter banks.

a highly non-linear problem space and only works well for relatively short filters (typically less than 14 taps). The new technique works quickly and effectively for filters of lengths up to 50 or more taps and provides considerable flexibility to the designer.

An alternative technique, which is based on a flat-delay allpass filter and is very neat, is given in [5], but this does not produce linear-phase complex wavelets and so is less attractive for image processing applications.

2. FILTER REQUIREMENTS FOR Q-SHIFT COMPLEX WAVELETS

The key properties required for the Q-shift filters are that they should provide a group delay of either $\frac{1}{4}$ or $\frac{3}{4}$ of a sample period, while also satisfying the usual 2-band filterbank constraints of no aliasing and perfect reconstruction. Since the filters are even-length, the time reverse of a $\frac{1}{4}$ sample delay filter is a $\frac{3}{4}$ sample delay filter, giving the required delay difference of $\frac{1}{2}$ sample if the filter and its time reverse are used in trees a and b respectively. This then leads to complex transform bases in which the imaginary part (from tree b) is the time reverse of the real part (from tree a), and so the complex bases exhibit conjugate symmetry (with a $45°$ rotation) about their mid points and hence have linear phase responses, even if the real and imaginary parts separately are *not* linear-phase.

Additionally we believe that it is advantageous in many emerging applications of complex wavelets that the Q-shift filters should be orthogonal (not just bi-orthogonal) to make the transform a tight frame which conserves energy from the signal to the transform domain. This has the further advantage that the same filters (and their time-reverses) can be used in the forward and inverse transforms.

The design problem can now be simplified to designing the four even-length filters of the 2-band filter banks of fig. 2 such that they provide:

1. No aliasing.
2. Perfect reconstruction
3. Orthogonality
4. Approximate group delay of $\frac{1}{4}$ sample for H_0.
5. Good smoothness properties when iterated over scale.

To achieve the first three of these we have the standard conditions:

$$G_1(z) = zH_0(-z)\,; \quad H_1(z) = z^{-1}G_0(-z) \tag{1}$$

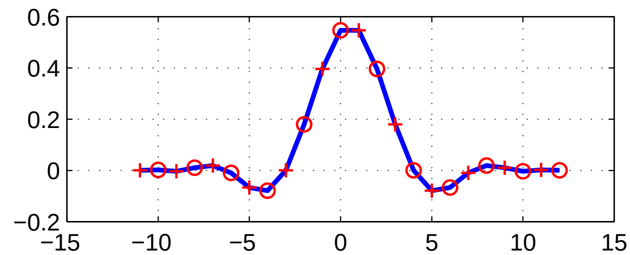

Fig. 3. Impulse response of $H_{L2}(z)$ for $n = 6$. The H_0 and G_0 filter taps are shown as circles and crosses respectively.

$$H_0(z)G_0(z) + H_0(-z)G_0(-z) = 2 \tag{2}$$

$$G_0(z) = H_0(z^{-1}) \tag{3}$$

where we assume that each filter is of length $2n$ taps and the H filter terms extend from z^{n-1} to z^{-n}, while the G filters extend from z^n to z^{1-n}.

To achieve the fourth property, we design a length $4n$ linear-phase (symmetric) lowpass filter, $H_{L2}(z)$, to operate at twice the required sampling rate, and then subsample it by 2:1, as shown in fig. 3. This filter will have a delay of $\frac{1}{2}$ sample, and, as long as it has negligible gain between $\frac{1}{4}$ and $\frac{1}{2}$ of its sample rate, then the subsampled filter, $H_0(z)$, will have half of its delay (ie $\frac{1}{4}$ sample) and twice its bandwidth. We define H_{L2} in terms of H_0 and G_0 as follows:

$$\begin{aligned} H_{L2}(z) &= H_0(z^2) + z^{-1}H_0(z^{-2}) \\ &= H_0(z^2) + z^{-1}G_0(z^2) \end{aligned} \tag{4}$$

Finally, to ensure the fifth property (smoothness), we must ensure that the stopband of $H_0(z)$ at each scale suppresses as much energy as possible at frequencies where the unwanted passbands appear from subsampled filters operating at coarser scales. It is sufficient to consider the combined frequency response of H_0 over just two scales, which is

$$H_0(z)\,H_0(z^2)|_{z=e^{j\omega}} = H_0(e^{j\omega})\,H_0(e^{2j\omega}) \tag{5}$$

If the passband and stopband of $H_0(z)$ cover $0 \le \omega \le \omega_p$ and $\omega_s \le \omega \le \pi$ respectively, and there is a transition band in between, then the unwanted transition and pass bands of $H_0(z^2)$ will extend from $\pi - \frac{\omega_s}{2}$ to π. If the stopband of $H_0(z)$ is to suppress these unwanted bands from $H_0(z^2)$ then

$$\omega_s \le \pi - \frac{\omega_s}{2} \qquad \therefore\ \omega_s \le \frac{2\pi}{3} \tag{6}$$

The perfect reconstruction and orthogonality conditions of equ. (2) and (3) require that $H_0(z)\,H_0(z^{-1})$ must have no terms in z^{2k} (k integer) except the term in z^0. This in turn implies that $H_0(z^2)\,H_0(z^{-2})$ must have no terms in z^{4k} except the term in z^0. It is then straightforward to use equ. (4) to show that $H_{L2}(z)\,H_{L2}(z^{-1})$ must have no terms in z^{4k} except the term in z^0. Hence we can include the perfect reconstruction requirement as a direct design constraint on $H_{L2}(z)\,H_{L2}(z^{-1})$ ($= H_{L2}^2(z)$, since H_{L2} is symmetric).

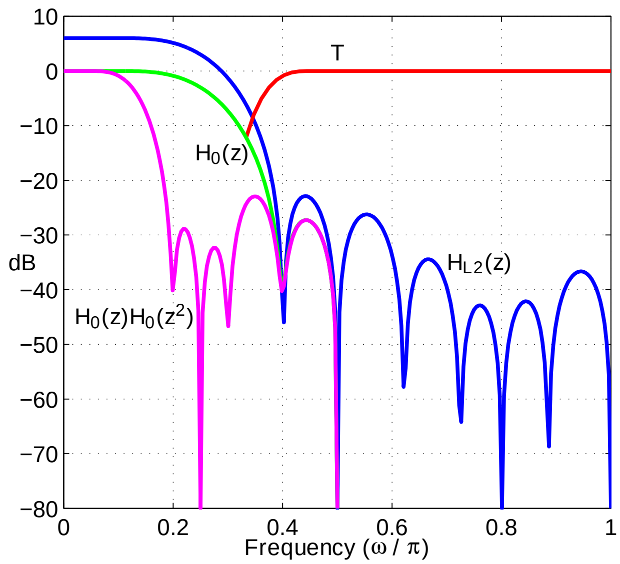

Fig. 4. Frequency responses of $H_{L2}(z)$ (blue), $H_0(z)$ (green), $H_0(z)\ H_0(z^2)$ (magenta), and the gain correction matrix $\mathbf{T}$ (red) for $n = 6$ (12 taps for H_0).

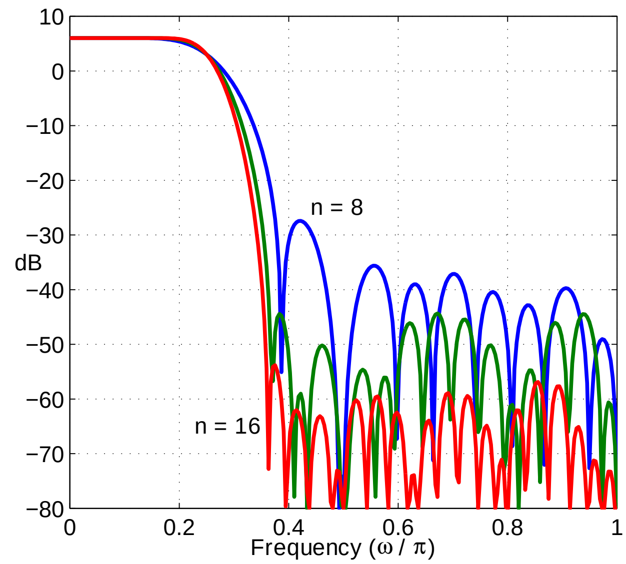

Fig. 5. Frequency responses of $H_{L2}(z)$ for $n = 8$ (blue), $n = 12$ (green) and $n = 16$ (red). Each filter has one predefined zero at $\omega = \frac{\pi}{2}$ and one at $\omega = \pi$.

3. OPTIMIZATION FOR MINIMUM SQUARED ERROR IN THE FREQUENCY DOMAIN

We have now reduced the ideal design conditions for the length $4n$ symmetric lowpass filter H_{L2} to be:

1. Zero amplitude for all the terms of $H_{L2}^2(z)$ in z^{4k} except the term in z^0, which must be unity;

2. Zero (or near-zero) amplitude of $H_{L2}(e^{j\omega})$ for the stopband, $\frac{\pi}{3} \leq \omega \leq \pi$.

Condition 1 is a set of quadratic constraints on the elements of the filter tap vector $\mathbf{h}_{L2}$, while condition 2 is a set of linear constraints on $\mathbf{h}_{L2}$, evaluated at a sufficiently fine set of frequencies covering the stopband. Together they form an overdetermined set of equations for the $2n$ unknowns that form one half of the symmetric vector $\mathbf{h}_{L2}$. If the constraints were all linear, the least mean square (LMS) error solution could be found in the standard way using the pseudo-inverse of the matrix which defines the equations.

To deal with the quadratic constraints, we take the straightforward approach of linearising the problem and using an iterative solution. If $\mathbf{h}_{L2}$ at iteration i is given by $\mathbf{h}_i = \mathbf{h}_{i-1} + \Delta\mathbf{h}_i$, then, since convolution ($*$) is commutative,

$$\begin{aligned} \mathbf{h}_i * \mathbf{h}_i &= (\mathbf{h}_{i-1} + \Delta\mathbf{h}_i) * (\mathbf{h}_{i-1} + \Delta\mathbf{h}_i) \\ &= \mathbf{h}_{i-1} * (\mathbf{h}_{i-1} + 2\Delta\mathbf{h}_i) + \Delta\mathbf{h}_i * \Delta\mathbf{h}_i \quad (7) \end{aligned}$$

If $\Delta\mathbf{h}_i$ is assumed to become small as i increases, the final term can be neglected and the convolution is expressed as a linear function of $\Delta\mathbf{h}_i$.

Hence the design problem can now be expressed as – solve for $\Delta\mathbf{h}_i$ such that:

$$\mathbf{C}\,(\mathbf{h}_{i-1} + 2\Delta\mathbf{h}_i) = [0 \ldots 0\ 1]^T \quad (8)$$

$$\mathbf{F}\,(\mathbf{h}_{i-1} + \Delta\mathbf{h}_i) \simeq [0 \ldots 0]^T \quad (9)$$

where $\mathbf{C}$ is a matrix which calculates every fourth term in the convolution with $\mathbf{h}_{i-1}$, and $\mathbf{F}$ is a matrix which evaluates the Fourier transform at M discrete frequencies ω from $\frac{\pi}{3}$ to π (typically $M \simeq 8n$ to ensure that all sidelobe maxima and minima are captured reasonably accurately). Note that only one side of the convolution is needed in $\mathbf{C}$, since the result is symmetric about the central term. Also, the columns of $\mathbf{C}$ and $\mathbf{F}$ can be combined in pairs so that only the first half of the symmetric $\Delta\mathbf{h}_i$ need be solved for.

In typical applications of complex wavelets, it is often more important to ensure high accuracy in the perfect reconstruction condition than to produce highly smooth wavelets. This why we have shown equ. (8) as an equality while equ. (9) is only an approximation. Within an LMS environment, we can produce high accuracy solutions to some equations by scaling these up by a factor, β_i, which is progressively increased with i. Hence we may now express the optimisation as the iterative LMS solution of:

$$\begin{bmatrix} 2\beta_i\mathbf{C} \\ \mathbf{F} \end{bmatrix} \Delta\mathbf{h}_i = \begin{bmatrix} \beta_i(\mathbf{c} - \mathbf{C}\,\mathbf{h}_{i-1}) \\ -\mathbf{F}\,\mathbf{h}_{i-1} \end{bmatrix} \quad (10)$$

$$\mathbf{h}_i = \mathbf{h}_{i-1} + \Delta\mathbf{h}_i \quad (11)$$

where $\mathbf{c} = [0 \ldots 0\ 1]^T$. Typically we use $\beta_i = 2^i$, and iterate over about 20 iterations, so that reconstruction errors are of the order of $2^{-20} \simeq 10^{-6}$ of smoothness errors.

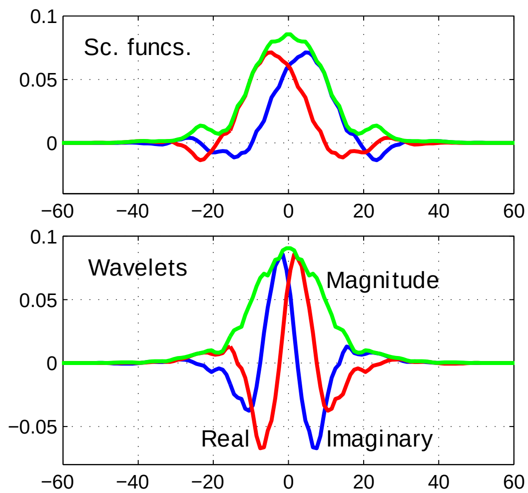

Fig. 6. Scaling functions and wavelets at level 4 for $n = 6$. Red and blue curves show the real and imaginary parts, and the green curves show the magnitudes of the complex waveforms.

There are two final refinements. One takes account of the transition band gain when evaluating the frequency domain errors. The other allows predefined zeros to be inserted at $z = -1$ ($\omega = \pi$) in either $H_0(z)$ or $H_{L2}(z)$.

To include transition band effects, we scale the rows of $\mathbf{F}$ by the gain of $H_0(z^2)/H_0(1)$ at frequencies corresponding to $\frac{\pi}{3} \leq \omega \leq \frac{\pi}{2}$ in the frequency domain of H_{L2}. Since H_{L2} has twice the sampling rate of the filter H_0, we must use the substitution $z = e^{2j\omega}$ in $H_0(z^2)$. For $\frac{\pi}{2} < \omega \leq \pi$ we use a unit scaling of the rows of $\mathbf{F}$ because all of these components must be kept small to avoid errors when we subsample H_{L2} by 2 to get H_0. The function which scales $\mathbf{F}$ is shown in red in fig. 4. This scaling down of $\mathbf{F}$ allows larger stopband gains at frequencies just above $\frac{\pi}{3}$ where the transition band of the next-level filter ($H_0(z^2)$ in equ. (5)) provides some additional attenuation.

To insert predefined zeros in $H_0(z)$ or $H_{L2}(z)$, we first note from equ. (4) that a zero at $z = e^{j\pi}$ in H_0 will be produced by a pair of zeros at $z = e^{\pm j\pi/2}$ in H_{L2}. We can force zeros in H_{L2} by forming a convolution matrix $\mathbf{H}_f$ such that $\mathbf{H}_f\ \mathbf{h}'_i = \mathbf{h}_i$, where $\mathbf{h}'_i$ is the coefficient vector of the filter which represents all the zeros of H_{L2} that are *not* predefined, $\mathbf{H}_f$ produces convolution with the predefined zeros, and $\mathbf{h}_i$ is the coefficient vector of H_{L2}, as before. $\mathbf{H}_f$ can be included in equ. (10,11), so that we now solve for $\Delta\mathbf{h}'_i$ at each iteration. We can also incorporate a diagonal gain matrix $\mathbf{T}_{i-1}$ to represent the transition band gain correction to get

$$\begin{bmatrix} 2\beta_i \mathbf{C} \\ \mathbf{T}_{i-1}\mathbf{F} \end{bmatrix} \mathbf{H}_f\ \Delta\mathbf{h}'_i = \begin{bmatrix} \beta_i(\mathbf{c} - \mathbf{C}\ \mathbf{h}_{i-1}) \\ -\mathbf{T}_{i-1}\mathbf{F}\ \mathbf{h}_{i-1} \end{bmatrix} \quad (12)$$

$$\mathbf{h}_i = \mathbf{h}_{i-1} + \mathbf{H}_f\ \Delta\mathbf{h}'_i \quad (13)$$

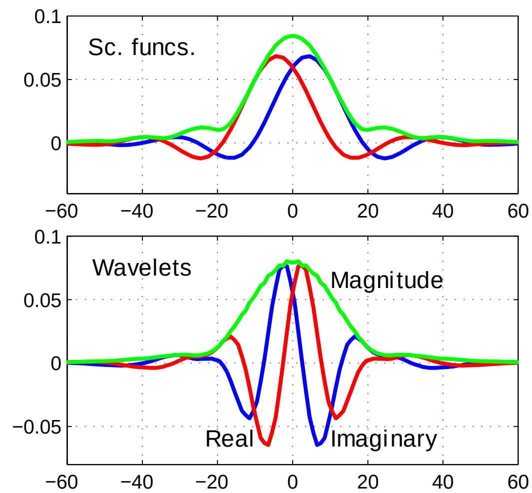

Fig. 7. Scaling functions and wavelets at level 4 for $n = 12$.

The incorporation of predefined zeros, allows the designer to control the amount of flatness in key regions of the frequency domain and gives additional design flexibility. H_{L2} always has at least one zero at $\omega = \pi$, because it is an even-length symmetric filter.

4. RESULTS

Figures 4 and 5 show the frequency responses of the optimized filter H_{L2} for $n = 6, 8, 12$ and 16. Note the improvement in stopband attenuation as n is increased. Figure 7 shows the improved smoothness of the resulting level-4 bases, when $n = 12$ compared with $n = 6$ (fig. 6). Many more results could have been shown, to illustrate the wide range of tradeoffs available with this design method and comparisons with other methods but space limitations do not permit this.

Papers by the author and Matlab code for this algorithm are available at http://www-sigproc.eng.cam.ac.uk/˜ngk/.

5. REFERENCES

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