# Least-squares migration using complex wavelets

*Mark Miller* and Nick Kingsbury, Cambridge University; Richard Hobbs, Durham University*

## Summary

The use of multi-scale decompositions has lead to advances in representation, restoration and reconstruction of images in many signal processing applications involving non-stationary data. In this paper, a complex wavelet based least-squares migration (LSM) method is proposed. The method is derived in a Bayesian framework and an example of the method is presented.

## Introduction

LSM has been shown to be effective in optimizing the reconstruction of subsurface reflectivity, particularly in cases of missing or undersampled data or uneven subsurface illumination (Nemeth, 1999; Duquet et al., 2000).

In standard LSM, the subsurface reflectivity model parameters are usually defined as a grid of point scatterers over the area to be migrated. We propose an approach to pre-stack LSM using the Dual Tree Complex Wavelet Transform (DT-CWT) as a basis for the reflectivity.

The DT-CWT is chosen for its key advantages compared to other wavelet transforms (Kingsbury, 2001). These are summarized as follows:

- Shift invariance
- Directional selectivity
- Perfect reconstruction
- Limited redundancy ($2^d$ where d is the dimension of the data)
- Efficient computation

A review of wavelet theory and concepts is given in Mallat (1999). More details about the DT-CWT are available in Kingsbury (2001).

The use of a complex wavelet transform with similar properties to the DT-CWT in seismic processing was suggested in Fernandes (2001). A wavelet basis was used for linear inversion with application to well logging in Li et. al., (1996), using a fully decimated wavelet transform and the Haar wavelet. However, the transform used does not have the shift invariance and directional selectivity properties of the DT-CWT. Wavelets have also been used in other linearized inverse scattering problems (Miller and Willsky, 1996).

Wavelet bases have a reputation for decorrelating or diagonalizing a range of non-stationary signals. This has lead to extensive use of wavelet bases in the area of information coding and compression. In LSM, diagonalization of the model space affords a more accurate but practical representation of prior information about the model parameters. This prior information is incorporated in the constraint term of the cost function that is minimized.

The constraint term becomes more important as the amount of recorded data is reduced relative to that to be reconstructed. Using a more sophisticated constraint becomes more important for missing or undersampled data problems or when increased resolution is required in the reflectivity model to be reconstructed. Alternatively, relaxing sampling requirements can reduce the cost of data acquisition. The constraint term is often ignored or has little attention paid to it in LSM literature. We will propose a method of definition of this term for a complex wavelet basis.

## Least-squares migration using complex wavelets

LSM relates the observed scattered acoustic data $\mathbf{d}$ to the reflectivity model $\mathbf{m}$ via the linear forward modeling operator $\mathbf{L}$ :

$$\mathbf{d} = \mathbf{L}\mathbf{m} + \mathbf{n} \quad (1)$$

where the additive noise $\mathbf{n}$ is Gaussian distributed with covariance $\mathbf{C_n}$ . At this point we perform a change of model parameters to the wavelet domain. Let $\mathbf{P}$ represent the inverse wavelet transform so that $\mathbf{m} = \mathbf{P}\mathbf{w}$ . Equation 1 becomes:

$$\mathbf{d} = \mathbf{L}\mathbf{P}\mathbf{w} + \mathbf{n} \quad (2)$$

The probability density function for the scattered data is then:

$$\begin{aligned} p(\mathbf{d} \mid \mathbf{w}) \\ &= \frac{1}{\sqrt{2\pi|\mathbf{C_n}|}} \exp\left( \frac{-(\mathbf{d}-\mathbf{LPw})^T \mathbf{C_n}^{-1} (\mathbf{d}-\mathbf{LPw})}{2} \right) \\ &\propto \exp\left( \frac{-(\mathbf{d}-\mathbf{LPw})^T \mathbf{C_n}^{-1} (\mathbf{d}-\mathbf{LPw})}{2} \right) \end{aligned} \quad (3)$$

The maximum a posteriori (MAP) estimate for the wavelet coefficients, which maximizes the posterior probability $p(\mathbf{w} \mid \mathbf{d})$ is:

$$\begin{aligned} \mathbf{w}_{\mathrm{MAP}} &= \arg\max_{\mathbf{w}} \ p(\mathbf{w} \mid \mathbf{d}) \\ &= \arg\max_{\mathbf{w}} \ p(\mathbf{d} \mid \mathbf{w})\, p(\mathbf{w}) \end{aligned} \qquad (4)$$

The second line of (4) follows from Bayes theorem. The probability $p(\mathbf{w})$ is the prior probability for the wavelet coefficients of the reflectivity data. Taking the negative logarithm yields the appropriate cost function in terms of the wavelet coefficients:

$$E(\mathbf{w}) = \left(\mathbf{d} - \mathbf{LPw}\right)^T \mathbf{C}_{\mathbf{n}}^{-1} \left(\mathbf{d} - \mathbf{LPw}\right) + f_{\mathbf{w}}(\mathbf{w}) \qquad (5)$$

where $f_{\mathbf{w}}(\mathbf{w}) = -\log(p(\mathbf{w}))$ is often referred to as the constraint term. It is proposed that the prior for the wavelet coefficients can be more accurately represented by the product of uncorrelated Gaussian distributions than that of the reflectivity model. In this case $f_{\mathbf{w}}(\mathbf{w}) = \mathbf{w}^T \mathbf{C}_{\mathbf{w}}^{-1} \mathbf{w}$, where $\mathbf{C}_{\mathbf{w}}$ is the covariance of the wavelet coefficients of the reflectivity signal and the cost function (5) is a quadratic function of the wavelet coefficients. We compare this to the cost function of the traditional LSM approach:

$$E(\mathbf{m}) = \left(\mathbf{d} - \mathbf{Lm}\right)^T \mathbf{C}_{\mathbf{n}}^{-1} \left(\mathbf{d} - \mathbf{Lm}\right) + f_{\mathbf{m}}(\mathbf{m}) \qquad (6)$$

Note that the Bayesian derivation above also produces the standard LSM cost function when no change of variables is performed. The wavelet coefficients that minimize equation 5 are:

$$\hat{\mathbf{w}}_{\mathrm{MAP}} = \left(\mathbf{P}^T \mathbf{L}^T \mathbf{C}_{\mathbf{n}}^{-1} \mathbf{LP} + \mathbf{C}_{\mathbf{w}}^{-1}\right)^{-1} \mathbf{P}^T \mathbf{L}^T \mathbf{C}_{\mathbf{n}}^{-1} \mathbf{d} \qquad (7)$$

The estimate for the wavelet coefficients is obtained in a similar way to traditional LSM by minimizing (5) using a conjugate gradient algorithm. The reflectivity estimate is then found by applying the inverse wavelet transform:

$$\hat{\mathbf{m}} = \mathbf{P}\hat{\mathbf{w}} \qquad (8)$$

Note that the forward and inverse transforms expressed using the matrices $\mathbf{P}^T$ and $\mathbf{P}$ are not implemented as matrix multiplications but using much faster wavelet decomposition and reconstruction algorithms.

The conjugate gradient minimization algorithm regards the real and imaginary parts of the complex wavelet coefficients as separate variables. Furthermore, the coefficients are assumed to be decorrelated resulting in a diagonal $\mathbf{C}_{\mathbf{w}}$ matrix. However, the variances of the real and imaginary parts are set to the same value but are allowed to vary within each subband, as in wavelet based inversion of de Rivaz and Kingsbury (2001).

One area where the use of a wavelet basis makes implementation of LSM more difficult is the preconditioning of the system. To speed up convergence, conjugate gradient algorithms used to solve linear inversion problems are usually preconditioned by scaling the variables prior to minimization. In its usual form the diagonal elements of the Hessian of the cost function $\nabla^2 E$ need to be calculated, so that they can be scaled to unity by the preconditioning. For standard LSM, ignoring the constraint term, the Hessian is:

$$\nabla^2 E = \mathbf{L}^T \mathbf{C}_{\mathbf{n}}^{-1} \mathbf{L} \qquad (9)$$

However, for wavelet LSM this becomes:

$$\nabla^2 E = \mathbf{L}^T \mathbf{P}^T \mathbf{C}_{\mathbf{n}}^{-1} \mathbf{PL} \qquad (10)$$

Calculating the diagonal elements of (10) is more difficult. Fortunately, for preconditioning the exact values are not required and a rough estimate is sufficient. This can be obtained using heavy interpolation and by being aware of the small support of most of the wavelet basis functions.

**Numerical Example**

Due to its computational expense, LSM is best suited to target oriented applications, where a smaller area or volume of interest is reconstructed after an initial reconnaissance using another faster migration technique (Jiang and Schuster, 2003). We will adopt this approach. Also, in the manner of Jiang and Schuster (2003), we use an otherwise migrated estimate for the reflectivity to initialize the LSM algorithm. This initial estimate is also used in determining the variances of the wavelet coefficients for the subsequent least-squares migration.

The synthetic data shown in figure 2 was generated using Seismic Unix (Cohen and Stockwell, 2003). This sparse dataset consists of six common shot gathers spaced at 300m with 30 traces in each gather. The maximum offset in each gather is 1500m. The velocity field is linearly increasing with depth from $1500\ \mathrm{ms}^{-1}$ at depth $z = 0$ at a rate of $0.8\ \mathrm{ms}^{-1}$ per meter. The reflectors used to generate the data are displayed in figure 1. Kirchhoff migration is used to obtain an initial estimate of the reflectivity. The Kirchhoff migrated section is shown in figure 3.

LSM with complex wavelets is now applied to the problem. The variances of the coefficients are obtained from the forward transform of the initial Kirchhoff estimate.

We compare this result to that obtained using non-wavelet least-squares migration cost function (6) with the constraint defined as $f_{\mathbf{w}}(\mathbf{w}) = \mathbf{m}^T \mathbf{C}_{\mathbf{m}}^{-1} \mathbf{m}$, where $\mathbf{C}_{\mathbf{m}} = \sigma_{\mathbf{m}}{}^2 \mathbf{I}$ and $\sigma_{\mathbf{m}}{}^2$ is the variance of the reflectivity model parameters, which is assumed to be constant. This constraint is used elsewhere, for example, in Duijndam et. al. (2000). As for the wavelet based conjugate gradient algorithm, preconditioning is used and the estimate is initialized using the Kirchhoff migration result.

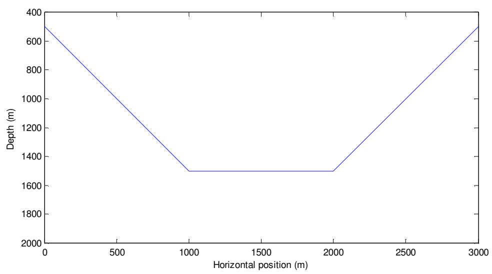


Figure 1: Reflector used to generate the synthetic data in figure 2

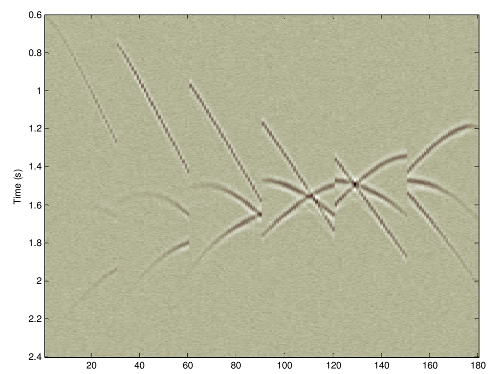


Figure 2: Synthetic seismogram of the reflector in figure 1

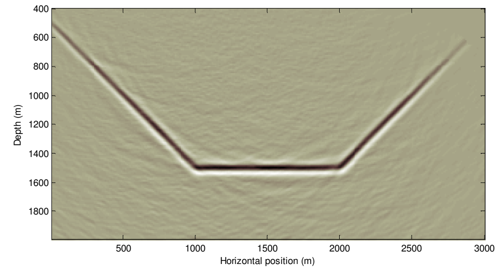


Figure 3: Kirchhoff migration of the data in figure 2

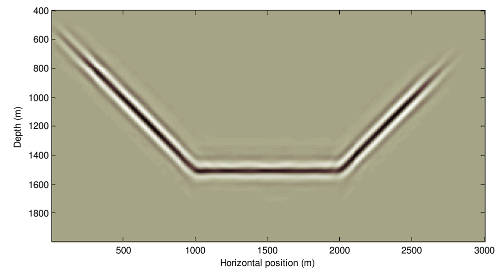


Figure 4: Inverted reflectivity model obtained after 10 iterations of LSM with complex wavelets

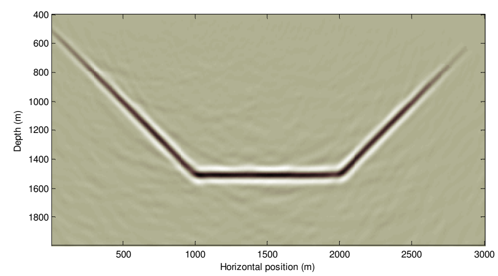


Figure 5: Inverted reflectivity after 10 iterations of non-wavelet LSM with $\mathbf{C}_{\mathbf{m}} = \sigma_{\mathbf{m}}{}^2 \mathbf{I}$

The complex wavelet LSM method in figure 4 shows improvement over both the Kirchhoff migration and the non-wavelet LSM method. This is to be expected, as the wavelet method uses a greater amount of prior information but it does demonstrate the potential of the method.

The method is now tested on reflectors at a range of dip angles. The acquisition geometry is as for the previous example but the different reflectors are displayed in figure 6. Figure 7 shows the Kirchhoff migrated initializing estimate and figure 8 the result of complex wavelet LSM. It is observed that complex wavelet LSM offers significant improvement for this sparsely sampled data set.

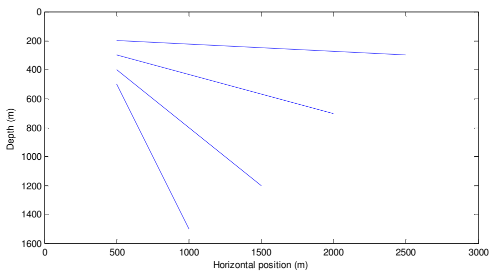


Figure 6: Reflectors at various dip angles

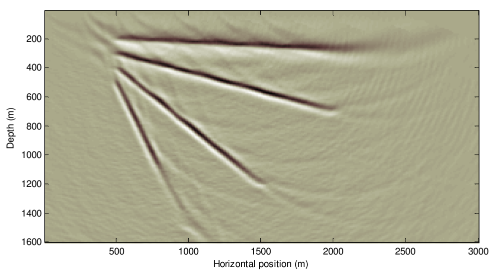


Figure 7: Kirchhoff migration result for reflectors at various dip angles

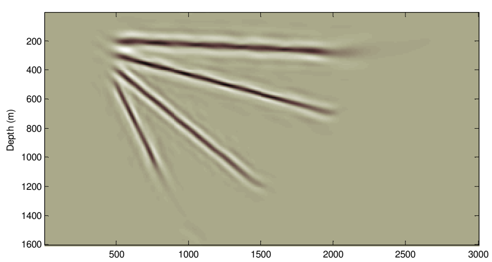


Figure 8: Inverted reflectivity model obtained after 10 iterations of LSM with complex wavelets

## Conclusions

A complex wavelet based LSM algorithm has been presented where minimization of the least-squares cost function is performed in the wavelet domain rather than the standard reflectivity model domain. A method for determining the prior probability that defines the constraint has been outlined that allows the statistical characteristics of the subsurface reflectivity to vary with locality as well as frequency.

## References

Cohen, J. K., and Stockwell, Jr. J.W., 2003, SWP/SU: Seismic Unix Release 37: a free package for seismic research and processing, Centre for Wave Phenomena, Colorado School of Mines

de Rivaz, P., and Kingsbury N. G., 2001, Bayesian Image Deconvolution abd Denoising using Complex Wavelets: Procedings of the IEEE International Conference on Image Processing, vol. 2, 273-276.

Diujndam, A. J. W., Volker, A. W. F., and Zwartjes, P. M., 2000, Reconstruction as an efficient alternative for least squares migration: 70th Ann. Internat. Meeting, Soc of Expl. Geophys., Expanded Abstracts, 1012-1015.

Duquet, B., Marfurt, K., and Dellinger, J., 2000, Kirchhoff modeling, inversion for reflectivity, and subsurface illumination: Geophysics, 65, 1195-1209.

Fernandes, F. C. A., Directional, 2001, Shift-Insensitive, Complex Wavelet Transforms with Controllable Redundancy: PhD Thesis, Rice University.

Jiang, Z., and Schuster, G., 2003, Target-oriented least squares migration: 73rd Ann. Internat. Meeting, Soc of Expl. Geophys., Expanded Abstracts, 1043-1046.

Kingsbury, N. G, 2001, Complex wavelets for shift invariant analysis and filtering of signals: J. of Applied and Computational Harmonic Analysis, vol. 10, no. 3, 234-253.

Li, X., Sacchi, M. D., and Ulrych, T. J., 1996, Wavelet transform inversion with prior scale information: Geophysics, 61, 1379-1385.

Mallat, S., 1999, A Wavelet Tour of Signal Processing: Academic Press.

Miller, E. L., and Willsky A. S., 1996, A mutiscale, statistically-based inversion scheme for linearized inverse scattering problems: IEEE Trans. on Geoscience and Remote Sensing, vol. 34, no. 2, 346-357.

Nemeth, T., Wu, C., and Schuster, G., 1999, Least-squares migration of incomplete reflection data: Geophysics, 64, 208-221.

## Acknowledgements

We would like to thank Dr Jerry Schuster and Min Zhou from the University of Utah for their assistance with our initial implementation of LSM and Rob Hardy from Trinity College, Dublin for his general assistance.