A Thresholded Landweber Algorithm for Wavelet-based Sparse Poisson Deconvolution

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***Abstract*—We propose a new iterative deconvolution algorithm for noisy Poisson images based on wavelet sparse regularization. To optimize the proposed cost function, we use a forward-backward splitting algorithm which has shown to find good solution for 3D microscopy deconvolution.**

I. Notations

In this document we use the following matrix/vector notations:

- $\mathbf{x}$ is a vector corresponding to the ground-truth image;
- $\mathbf{H}$ is a matrix representing the blurring operator;
- $\mathbf{b}$ is a constant vector representing the background signal;
- $\mathbf{y}$ is a random vector modeling the measurements;
- k indicates the number of iterations;
- $\mathbf{M}$ is the inverse wavelet transform whose columns are wavelet basis;
- $\mathbf{w}$ is a vector representing the wavelet coefficients;
- $\mathbf{Q}$ is a matrix representing observations without poisson noise;
- τ, β and α are regularization parameters

II. Algorithm

The algorithm we describe here is an iterative procedure which consists of a Poisson denoising stage proposed in [1] and a thresholded Landweber step [2], [3]. The key steps of our algorithm can be summarized as

Algorithm 1 Proposed Image Deconvolution Algorithm

1: **Inputs**: $\mathbf{H}, \mathbf{y}, \mathbf{b}, \mathbf{M}, \mathbf{w}_0, \mathbf{Q}_0, \tau$, β and α.
2: **while** iterations k **do**
3: $\quad \mathbf{Q}_{k+1} = \frac{1}{2}\left[\mathbf{Q}_k - \frac{1}{\beta} + \sqrt{\left(\mathbf{Q}_k - \frac{1}{\beta}\right)^2 + \frac{4\mathbf{y}}{\beta}}\right]$
4: $\quad \mathbf{v}_k = \mathbf{w}_k + \frac{1}{\alpha}\mathbf{M}^T\mathbf{H}^T\left(\mathbf{Q}_{k+1} - \mathbf{b} - \mathbf{H}\mathbf{M}\mathbf{w}_k\right)$
5: $\quad \mathbf{w}_{k+1} = \operatorname{sign}(\mathbf{v}_k)\max\left(|\mathbf{v}_k| - \frac{\tau}{\alpha}, 0\right)$
6: **end while**
7: **Output** deblurred image $\mathbf{x} = \mathbf{M}\mathbf{w}_{k+1}$

III. Variational interpretation

Let us begin with the general Poisson noise model as:

$$\mathbf{y} \sim \mathcal{P}(\mathbf{A}\mathbf{x} + \mathbf{b}) \tag{1}$$

where $\mathcal{P}(\boldsymbol{\lambda})$ is a Poisson-distributed random vector of mean $\boldsymbol{\lambda}$. Minimizing (1) with respect to $\mathbf{x}$ is equivalent to minimizing $-\log p(\mathbf{y}|\mathbf{x})$, such that

$$J_L(\mathbf{x}, \mathbf{y}) = \mathbf{1}^T(\mathbf{H}\mathbf{x} + \mathbf{b}) - \mathbf{y}^T \log(\mathbf{H}\mathbf{x} + \mathbf{b}) \tag{2}$$

A popular algorithm to minimize (2) is the Richardson–Lucy (RL) algorithm [4], [5]. However it is noted that the RL algorithm is not sufficient to prevent noise amplification during the deconvolution process due to the ill-posedness of this problem [1]. To overcome this, several authors propose to use explicit priors on the solution [1]. Here we propose a new cost function:

$$J(\mathbf{w}, \mathbf{Q}) = \frac{1}{2}\|\mathbf{Q} - \mathbf{b} - \mathbf{H}\mathbf{M}\mathbf{w}\|_2^2 + \beta\left(\mathbf{1}^T(\mathbf{Q}) - \mathbf{y}^T\log(\mathbf{Q})\right) + \tau\|\mathbf{w}\|_1 \tag{3}$$

Note that here we impose an l_1-norm prior in the wavelet domain due to the fact that natural image can be well sparsified using wavelet basis. The term $\|\mathbf{Q} - \mathbf{b} - \mathbf{H}\mathbf{M}\mathbf{w}\|_2^2$ is useful because it measures the residual in the observation data, which will effectively prevent the noise amplification during the deconvolution stage.

To optimize (3), we propose the following two steps:

$$\textbf{Step 1}: \mathbf{Q}_{k+1} = \arg\min_{\mathbf{Q}} J(\mathbf{w}_k, \mathbf{Q}); \tag{4}$$

$$\textbf{Step 2}: \mathbf{w}_{k+1} = \arg\min_{\mathbf{w}} J(\mathbf{w}, \mathbf{Q}_{k+1}); \tag{5}$$

Assuming $\mathbf{Q}_k$ is a sufficient estimate of $\mathbf{b} + \mathbf{H}\mathbf{M}\mathbf{w}$, we can optimize (4) via

$$\mathbf{Q}_{k+1} = \arg\min_{\mathbf{Q}} \frac{1}{2}\|\mathbf{Q} - \mathbf{Q}_k\|_2^2 + \beta\left(\mathbf{1}^T(\mathbf{Q}) - \mathbf{y}^T\log(\mathbf{Q})\right) \tag{6}$$

where the optimal solution can be found via [1]:

$$\mathbf{Q}_{opt} = \frac{1}{2}\left[\mathbf{Q}_k - \frac{1}{\beta} + \sqrt{\left(\mathbf{Q}_k - \frac{1}{\beta}\right)^2 + \frac{4\mathbf{y}}{\beta}}\right] \tag{7}$$

Optimizing (5) is equivalent to

$$\mathbf{w}_{k+1} = \arg\min_{\mathbf{w}} \frac{1}{2}\|\mathbf{Q}_{k+1} - \mathbf{b} - \mathbf{H}\mathbf{M}\mathbf{w}\|_2^2 + \tau\|\mathbf{w}\|_1 \tag{8}$$

This is a standard wavelet-domain regularization problem, and there are many techniques such as iterative soft thresholding (IST) [2], [3]:

$$1)\ \ \mathbf{v}_k = \mathbf{w}_k + \frac{1}{\alpha}\mathbf{M}^T\mathbf{H}^T\left(\mathbf{Q}_{k+1} - \mathbf{b} - \mathbf{H}\mathbf{M}\mathbf{w}_k\right) \tag{9}$$

$$2)\ \ \mathbf{w}_{k+1} = \operatorname{sign}(\mathbf{v}_k)\max\left(|\mathbf{v}_k| - \frac{\tau}{\alpha}, 0\right) \tag{10}$$

where α is a regularization parameter that can be optimized for every wavelet subband. As a result, we obtain the updating rules shown in Section II.

IV. Choice of the parameters

The parameters that needs to be adjusted for the proposed algorithm are k, τ, β and α. To ensure the convergence, the parameter α must satisfy $\alpha > \rho(\mathbf{M}^T\mathbf{H}^T\mathbf{H}\mathbf{M})$. In the experiment, we set regularization parameter $\tau = 10^{-2}$. k is chosen based on the standard stopping criteria, e.g., $\|\mathbf{x}_{k+1} - \mathbf{x}_k\| < \epsilon$, where ϵ and δ are fixed thresholds. We adjust the regularization parameter β to give the result that is most pleasing visually. For the wavelet basis, we choose the dual-tree complex wavelet transform because it has a good frequency selectivity and is almost shift-invariant [6].

References

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