IB Paper 6: Signal and Data Analysis

Handout 3: Fourier Transforms

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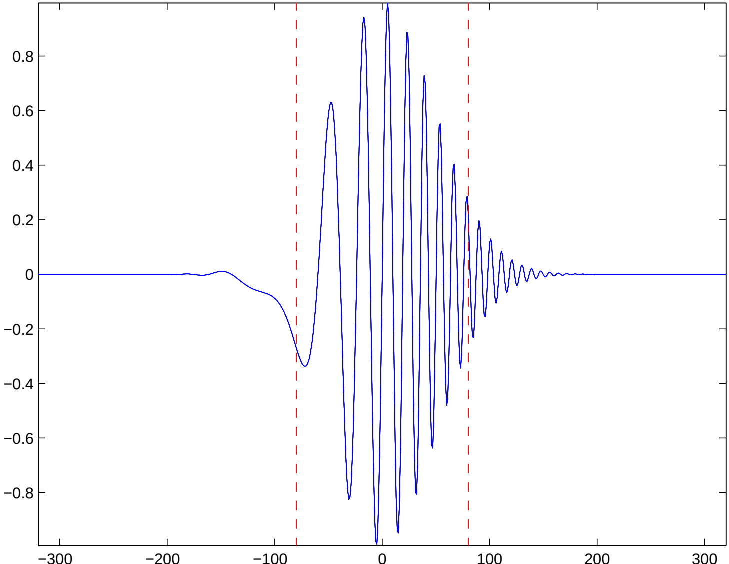
Fourier Transforms

- The Fourier Series is a useful tool for analysing periodic signals.
- However, this is quite restrictive, since most real-world signals are *not* periodic
- The topic of Spectrum Analysis attempts to determine the frequency content of general non-periodic signals.
- The theory of Fourier Transforms shows how to achieve this.
- Use Fourier series as a starting point and generalise to non-periodic signals.

Intuitive approach

Consider for example the non-periodic signal below:



We can make a periodic signal by extracting the centre part and periodically repeating it:

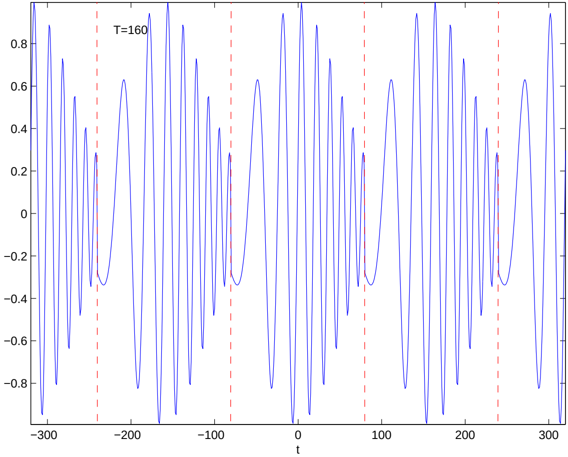


Figure 2 : Periodic repetition with $T = 160$

Now calculate the Fourier series for the periodically repeated signal and plot the coefficient magnitudes on a graph of amplitude vs. frequency. The fundamental frequency is $\omega_0 = 2\pi/T = 0.0393$rads/s, so we place the components at $n\omega_0$:

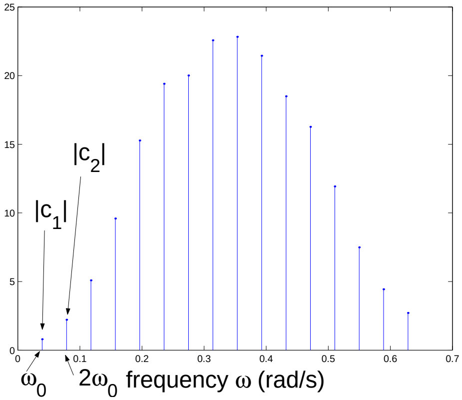


Figure 3 : Fourier coefficients for periodically repeated signal.

What happens if we repeat this procedure, each time increasing the period T and including more of the non-periodic signal?

- Fundamental frequency $\omega_0 = 2\pi/T$ decreases, so frequency components get closer and closer together
- As $T \rightarrow \infty$ we have a frequency component at *every* frequency ω.
- This is the idea behind the Fourier Transform; see figure 4 to see how the Fourier series gradually converges to give a *continuous* coverage of the frequency axis.

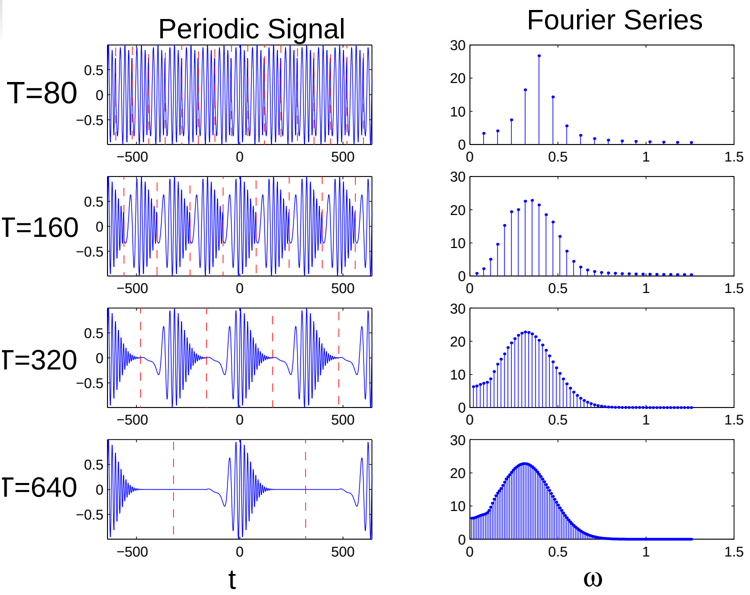


Figure 4 : Fourier series as $T \to \infty$.

Mathematical formulation

- Start with the complex Fourier series coefficient formula for a function $f(t)$:

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\mathrm{e}^{-j\omega_0 nt}dt \tag{1}$$

with $\omega_0 = 2\pi/T$, as before.

- c_n is the Fourier series component with frequency $n\omega_0$. Now, define some function of frequency $D_T(\omega)$ which equals Tc_n at the points $\omega = n\omega_0$, i.e.

$$D_T(n\omega_0) = Tc_n, \quad n = -\infty, ..., -1, 0, 1, ..., +\infty \tag{2}$$

D_T therefore looks like:

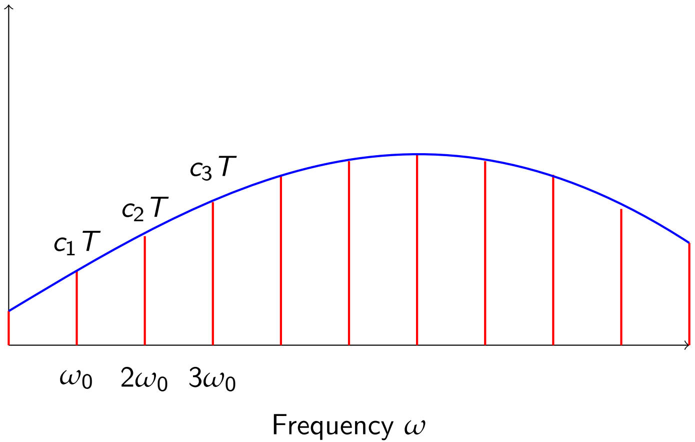


Figure 5 : The function $D_T(\omega)$ (c_n are shown as real-valued)

- Hence we can write

$$D_T(n\omega_0) = Tc_n = \int_{-T/2}^{T/2} f(t)e^{-j\omega_0 nt}dt \tag{3}$$

- A suitable possibility for the function $D_T(\omega)$ which satisfies the constraint of (3) is simply obtained by replacing $n\omega_0$ with a continuous variable ω in the above formula:

$$D_T(\omega) = \int_{-T/2}^{T/2} f(t)e^{-j\omega t}dt \tag{4}$$

- As we increase the period T, so ω_0 decreases, and the discrete points $D_T(n\omega_0)$ define the whole function $D_T(\omega)$ for all values of ω.

- The Fourier Transform $F(\omega)$ is obtained as the limit of $D_T(\omega)$ as $T \to \infty$, i.e.

$$
\begin{aligned}
F(\omega) = \operatorname*{Lim}_{T\to\infty} D(\omega) &= \lim_{T\to\infty} \int_{-T/2}^{T/2} f(t) e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt
\end{aligned}
$$

The Fourier transform is often referred to as the Fourier spectrum of a signal.

Comment: why is there a factor of T in the derivation? (See (3)). This serves to ‘normalise’ the integral - otherwise the integral would be zero for most functions!

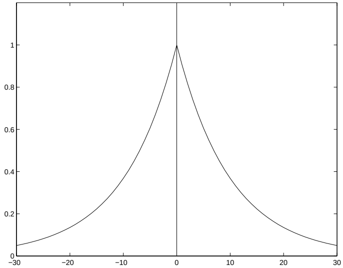
Interpretation of Fourier Transform

- The Fourier Transform is one of the most important and useful tools of signal analysis and mathematics in general.
- The Fourier series showed how to split a *periodic* signal into its constituent frequency components at ω_0, $2\omega_0$, ...
- Since the Fourier transform is derived as a limiting case of a Fourier series ($T \to \infty$), we can interpret it in a similar way. With the Fourier transform, however, we can analyse *non-periodic* functions and obtain their frequency components at *any frequency* ω .
- Thus the Fourier transform is a much more generally applicable tool.

Example

Consider the 'double exponential' function (with $a > 0$):

$$f(t) = \begin{cases} e^{-at} & t \geq 0 \\ e^{+at} & t < 0 \end{cases}$$



Calculate the Fourier transform of $f(t)$:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \int_{-\infty}^{0} e^{at}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t}dt + \int_{0}^{\infty} e^{(-a-j\omega)t}dt = \frac{2a}{a^2+\omega^2}$$

The following figure shows the Fourier coefficients $c_n T$ for 3 increasing values of T. The bottom plot has the Fourier transform overlayed as the continuous line - an almost perfect match.

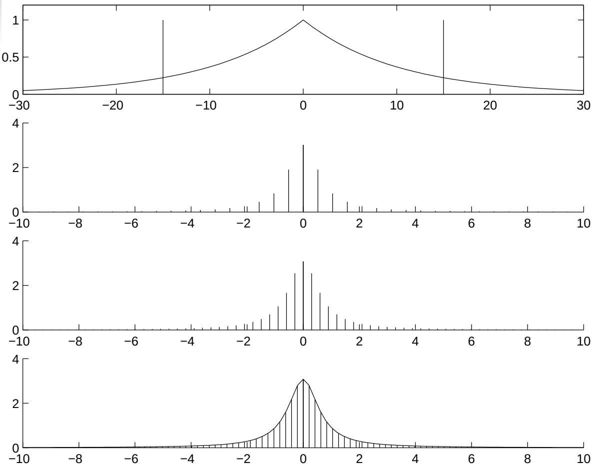


Figure 6 : (from top to bottom) a) the original function b) the Fourier coefficients, $c_n T$, when interval taken is $[-6, 6]$, c) coefficients for $[-10.5, 10.5]$, d) coefficients for $[-15, 15]$, overlayed with the Fourier transform of $f(t)$ (solid line)

The Inverse Fourier Transform (IFT)

It is possible to invert the Fourier transform to determine uniquely the signal $f(t)$ from its Fourier Transform $F(\omega)$. We derive the required formula using results for δ-functions from Handout 1.
Start from the Fourier transform formula:

$$F(\omega) = \int_{-\infty}^{\infty} f(t')e^{-j\omega t'}dt' \tag{5}$$

where a new time variable t' is used for convenience.
Multiply $F(\omega)$ by $e^{j\omega t}$ and integrate wrt ω:

$$I = \int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega = \int_{\omega=-\infty}^{\infty}\int_{t'=-\infty}^{\infty} f(t')e^{-j\omega t'}e^{j\omega t}\,dt'\,d\omega \tag{6}$$

Now change order of integration

$$I = \int_{t'=-\infty}^{\infty} f(t') \left\{ \int_{\omega=-\infty}^{\infty} e^{j\omega(t-t')} d\omega \right\} dt' \qquad (7)$$

Recall in Handout 1 we showed the result

$$\int_{-\infty}^{\infty} e^{j\omega\tau} d\omega = 2\pi\delta(\tau). \qquad (8)$$

Using this result with $\tau = t - t'$:

$$I = 2\pi \int_{-\infty}^{\infty} f(t')\delta(t-t')dt' = 2\pi f(t) \qquad (9)$$

where the final integral is done using the sifting property of δ-functions, as revised in Handout 1

Now rearrange to get:

$$f(t) = \frac{I}{2\pi} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

This is a unique expression for $f(t)$ in terms of $F(\omega)$, the Inverse Fourier Transform.

The forward and inverse **Fourier transform pair** can now be summarised as (see EI data book):

Fourier Transform

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt} \tag{10}$$

$$\tag{11}$$

Inverse Fourier Transform

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega} \tag{12}$$

Note - you may encounter some slightly different definitions of the Fourier transform. For example, if $F(\omega)$ is redefined as $F(\omega)/\sqrt{2\pi}$ then both the forward and inverse transforms have a factor of $1/\sqrt{2\pi}$ before the integral and the transform is 'symmetric'.

But these other definitions make no difference whatsoever to the principle of operation. In this course we will stick with the definitions given in equations (10) and (12).

Some Important Fourier Transforms

[Most of these are also given in the EI data book]

Complex exponential

$f(t) = \exp(j\omega_0 t)$ for a fixed frequency ω_0.

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t} dt.$$
$$= 2\pi\delta(\omega - \omega_0)$$

Once again, this integral follows from result in Handout 1.

As expected, all of the frequency content of $f(t) = \exp(j\omega_0 t)$ is concentrated on the frequency ω_0 as a δ-function.

Check result using inverse Fourier Transform:

$$
\begin{aligned}
f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \\
&= \int_{-\infty}^{\infty} \delta(\omega - \omega_0)e^{j\omega t}d\omega \\
&= e^{j\omega_0 t}
\end{aligned}
$$

as required.

Cosine

$f(t) = \cos(\omega_0 t)$.

Writing $\cos(\omega_0 t) = \frac{1}{2}(\exp(j\omega_0 t) + \exp(-j\omega_0 t))$ we obtain the result immediately from the Fourier transform of $\exp(j\omega_0 t)$:

$$F(\omega) = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] \tag{13}$$

This follows because the Fourier Transform is linear, i.e. for two functions $f_1(t)$ and $f_2(t)$:

$$\mathrm{FT}\{af_1(t) + bf_2(t)\} = aF_1(\omega) + bF_2(\omega)$$

[you can prove this for yourself from the definition of $F(\omega)$].

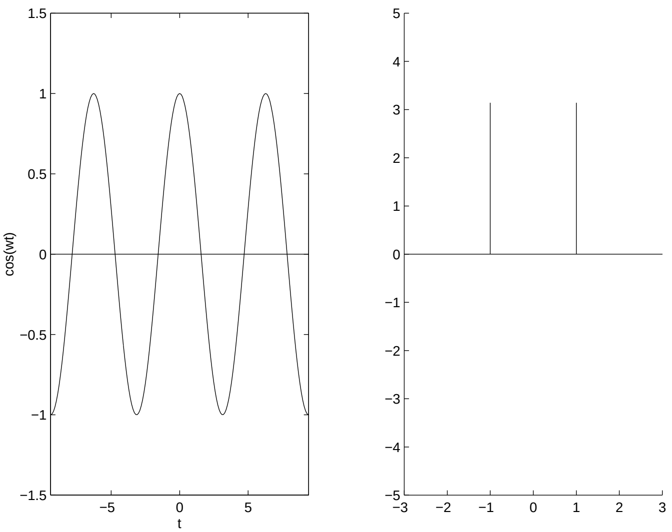


Figure 7 : Fourier transform of cosine

Sine.

$f(t) = \sin(\omega_0 t)$.

Using the same method as for cosine:

$$F(\omega) = j\pi(\delta(\omega + \omega_0) - \delta(\omega - \omega_0)) \tag{14}$$

δ-function

$f(t) = a\delta(t)$, where a is a constant. Its Fourier transform is

$$F(\omega) = \int_{-\infty}^{\infty} a\delta(t)\mathrm{e}^{-j\omega t}dt = a \qquad (15)$$

i.e. a constant for all frequencies.

DC offset

$f(t) = a$, where a is a constant. Fourier transform is

$$F(\omega) = a\int_{-\infty}^{\infty} \mathrm{e}^{-j\omega t}dt = a2\pi\delta(-\omega) = a2\pi\delta(\omega) \qquad (16)$$

Rectangular pulse

Consider the rectangular pulse

$$f(t) = \begin{cases} b & \text{for } -T/2 < t < T/2 \\ 0 & \text{otherwise} \end{cases}$$

Its Fourier transform is

$$F(\omega) = \int_{-\frac{T}{2}}^{\frac{T}{2}} b\mathrm{e}^{-j\omega t} dt = -\frac{b}{j\omega} \left[\mathrm{e}^{-j\omega t} \right]_{-\frac{T}{2}}^{\frac{T}{2}}$$

$$= \frac{2b}{\omega} \sin(\omega T/2) = bT \mathrm{sinc}\left(\frac{\omega T}{2}\right).$$

The Gaussian

Take a 'Gaussian' shaped pulse:

$$f(t) = \mathrm{e}^{-a^2 t^2}$$

The Fourier transform of this is:

$$F(\omega) = \frac{\sqrt{\pi}}{a}\mathrm{e}^{-\frac{\omega^2}{4a^2}}$$

In words, the Fourier transform of a Gaussian shape is itself a Gaussian shape - see probability and statistics course where this result is derived.

S. Godsill (2015)