IB Paper 6: Signal and Data Analysis

Handout 4: Properties of the Fourier Transform

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Properties of the Fourier Transform

We recall that if we knew the Fourier coefficients for some standard functions we were able to find the Fourier coefficients for other functions related to the original by *shifts, scalings, differentiation, etc.*. We now want to look at the corresponding properties of the Fourier transform

See also EI Data book for a fairly complete list of properties.

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where a and b are scalar constants. This is really just saying that integration is a linear operator - can check this result for yourself. This means we can find the Fourier transform of complicated functions by decomposing them as a sum of simpler functions ($f(t) = af_1(t) + bf_2(t)$) and then summing the transforms of the simpler functions ($F(\omega) = aF_1(\omega) + bF_2(\omega)$).

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$$f(\alpha t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\{ \frac{1}{\alpha} F\left(\frac{\omega'}{\alpha}\right) \right\} e^{j\omega' t} d\omega'$$

Studying the form of this equation, we can see that it is precisely in the form of an inverse Fourier transform, where the frequency function is $\frac{1}{\alpha}F\left(\frac{\omega'}{\alpha}\right)$. Hence:

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(This results also holds for $\alpha < 0$ – you can check this is the case).

Example

Recall the example of the rectangular pulse near the end of Handout 3.

The formula for the pulse is given by

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Thus $f(2t)$, which is a pulse of width $T/2$, will have FT

$$\frac{bT}{2} \operatorname{sinc}\left(\frac{\omega T}{4}\right)$$

thus producing a spectrum of double the width.

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> *If any function $f(t)$ has time duration T, and its Fourier transform $F(\omega)$ has frequency bandwidth B, then,*
>
> $TB \geq 1$ *(Time-Bandwidth product)*

This can be proved quite elegantly, but we won't go into the details here.

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You get to prove this for yourself - see examples paper 6/6 question 4.

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This result is fundamental to the **modulation** of signals (see Communications course - 2nd half of Lent Paper 6).

From the above modulation theorem we can deduce the FT of $f(t)\cos(\omega_0 t)$ in terms of the FT of $f(t)$, since

$$f(t)\cos(\omega_0 t) = \frac{1}{2}\{e^{j\omega_0 t}f(t) + e^{-j\omega_0 t}f(t)\}$$

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Hence, by linearity of the FT:

$$\boxed{f(t)\cos(\omega_0 t) \xleftrightarrow{FT} \frac{1}{2}F(\omega-\omega_0) + \frac{1}{2}F(\omega+\omega_0)} \qquad (5)$$

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The result is that the spectrum of the envelope is separated into two parts, each half of its original strength, and shifted along the ω axis by $\pm\omega_0$.

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Then, it is clear that repeated differentiation will simply bring down further factors of $(j\omega)$ – which leads to the result

$$f^{(n)}(t) \xleftrightarrow{FT} (j\omega)^n F(\omega) \tag{7}$$

where $f^{(n)}(t)$ denotes the nth derivative of $f(t)$

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Rearranging this gives:

$$2\pi f(-\omega') = \int_{-\infty}^{\infty} g(t')e^{-j\omega' t'}dt'$$

where the RHS is exactly the FT of $g(t')$.

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then we automatically have (without any integration) the *dual* Fourier transform pair:

$$g(t) \xleftrightarrow{FT} 2\pi f(-\omega)$$

This can be difficult to understand at first read-through. See also Example Sheet 6/6, Q4.

Example:

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Hence, by duality, the Fourier transform of $bT\mathrm{sinc}(tT/2)$ is a rectangle pulse with height $2b\pi$ and width T

The Multiplication Theorem and Parseval's Theorem

Consider two functions $f_1(t)$ and $f_2(t)$ with FTs $F_1(\omega)$ and $F_2(\omega)$ and look at the integral of the product of f_1 and f_2^*:

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Note the similarity between this and Parseval's theorem for Fourier series – here it is telling us that the amount of 'energy' in a system can be found either by integrating in the time domain or (equivalently) in the spectral domain.

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cf. similar result for Laplace Transforms.

We now show this. If $h(t) = f * g$, taking the FT of the convolution gives

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We now show this. If $h(t) = f * g$, taking the FT of the convolution gives

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Also, by duality (check this this for yourself):

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$$\boxed{2\pi f(t)g(t) \xleftrightarrow{FT} F(\omega) * G(\omega)} \tag{12}$$

Therefore we have the **VERY IMPORTANT** result that convolution in one domain is equivalent to multiplication in the other domain.

Fourier Transforms and Linear Systems

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Let $h(t)$ be the response of a linear system to an impulse $\delta(t)$ (the 'impulse response').

Consider an input to this system of $x(t)$. Since it is a linear system the output, $y(t)$, is the convolution $x(t) * h(t)$:

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which is the analogue of the result from Laplace transforms – the transfer function $H(\omega)$ is the FT of the impulse response and gives the ratio of the FT of the output to the FT of the input.

Fourier transform vs. Laplace transform

Compare the definitions:

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t}\, dt \qquad \bar{f}(s) = \int_{0}^{+\infty} f(t) e^{-st}\, dt$$

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and the two are equivalent (provided both integrals exist).

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Then the frequency response of the system can be calculated using either Fourier transforms or Laplace transforms since

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Note that standard 1A ac circuit theory is a simple application of Fourier transforms and frequency response to a differential equation system.

S. Godsill (2011), J. Lasenby (2009)