# 1B Paper 6: Communications

## Handout 4: Digital Baseband Modulation

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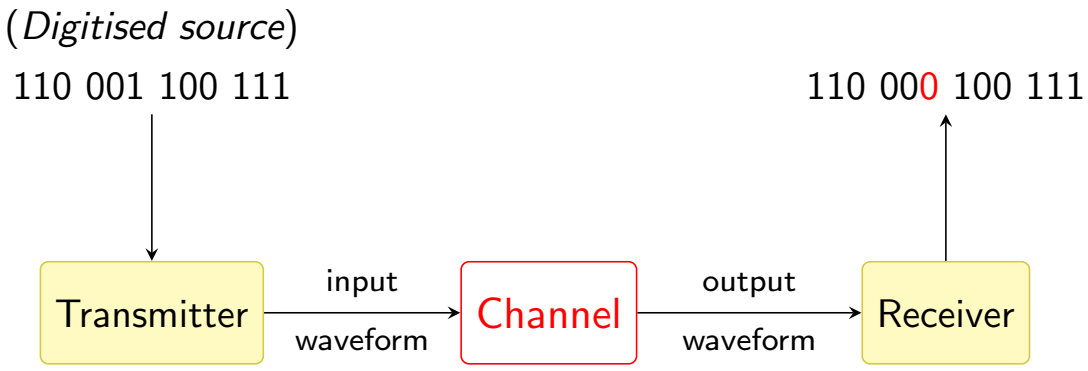
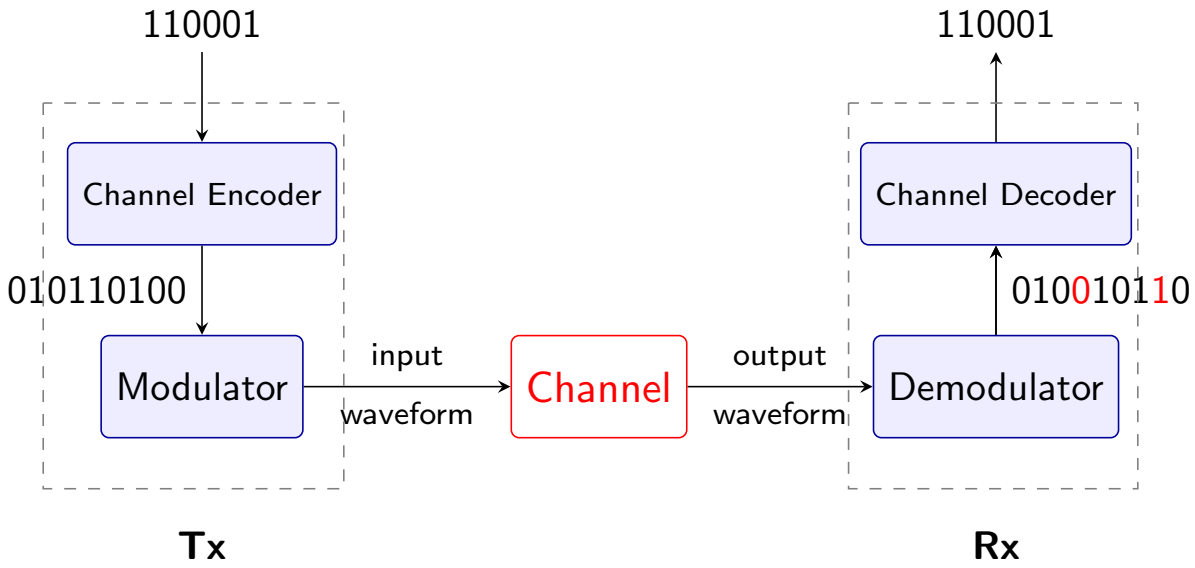
## Data Transmission

We have seen how analogue sources can be digitised. E.g., An MPEG or QuickTime file is a stream of bits

$\longleftrightarrow \quad \ldots 10110010001101010 \ldots$

Now we have to transport those bits across a channel:






The transmitter (Tx) does two things:

1. *Encoding*: Adding redundancy to the source bits to protect against noise
2. *Modulation*: Transforming the coded bits into waveforms.

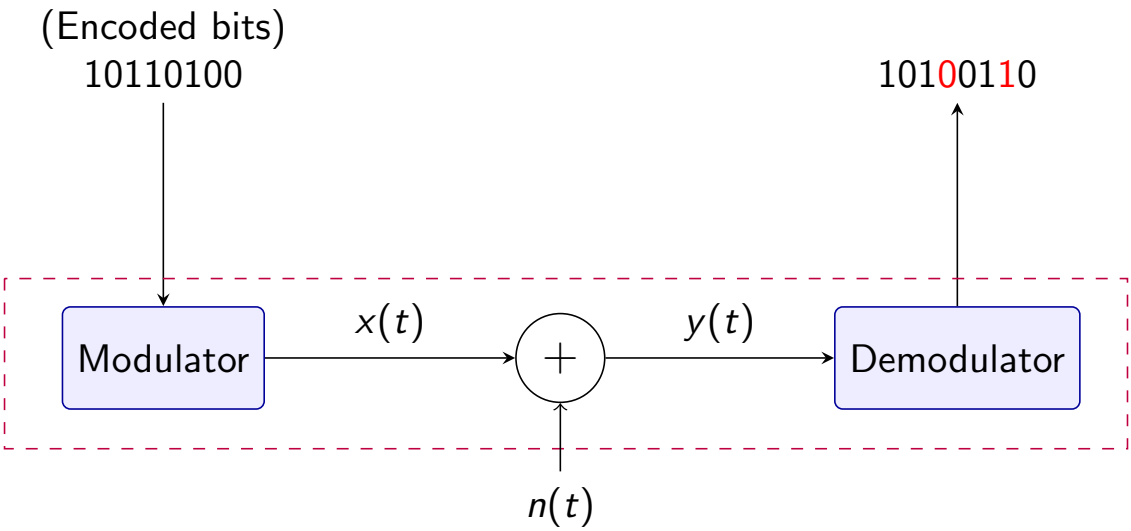
The receiver (Rx) does:

- *Demodulation*: noisy output waveform $\longrightarrow$ output bits
- *Decoding*: Try to correct errors in the output bits and recover the source bits



## Modulation/Demodulation

We'll first consider the modulation and demodulation blocks assuming that the channel encoder/decoder are fixed, and look at the design of the channel encoder and decoder later.




We now study a digital baseband modulation technique called *Pulse Amplitude Modulation* (PAM) & analyse its performance over an Additive White Gaussian Noise (AWGN) channel

## The Symbol Constellation

The digital modulation scheme has two basic components.

1. The first is a mapping from bits to real/complex numbers, e.g.

$$0 \to -A, \quad 1 \to A \quad \text{(binary symbols)}$$
$$00 \to -3A, \quad 01 \to -A, \quad 10 \to A, \quad 11 \to 3A \quad \text{(4-ary symbols)}$$

The set of values the bits are mapped to is called the *constellation*, e.g., the 4-ary constellation above is $\{-3A, A, A, 3A\}$.

Once we fix a constellation, a sequence of bits can be uniquely mapped to constellation symbols. E.g., with constellation $\{-A, A\}$

$$0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0 \longrightarrow -A,\ A, -A,\ A,\ A,\ A, -A, -A,\ A, -A$$

With constellation $\{-3A, -A, A, 3A\}$, the same sequence of bits is mapped as

$$\underline{01}\ \underline{01}\ \underline{11}\ \underline{00}\ \underline{10} \longrightarrow -A,\ -A,\ 3A,\ -3A,\ A$$

In a constellation with $M$ symbols, each symbol represents $\log_2 M$ bits



## The Pulse Shape

2. The second component of Pulse Amplitude Modulation is a unit-energy *baseband* waveform denoted $p(t)$, called the *pulse shape*. E.g., a sinc pulse or a rect pulse:

$$p(t) = \frac{1}{\sqrt{T}} \operatorname{sinc}\left(\frac{\pi t}{T}\right) \quad \text{or} \quad p(t) = \begin{cases} \frac{1}{\sqrt{T}} & \text{for } t \in (0, T] \\ 0 & \text{otherwise} \end{cases}$$

$T$ is called the *symbol* time of the pulse

A sequence of constellation symbols $X_0, X_1, X_2, \ldots$ is used to generate a *baseband* signal as follows

$$x_b(t) = \sum_k X_k p(t - kT)$$

Thus we have the following **important** steps to associate bits with a baseband signal $x_b(t)$:

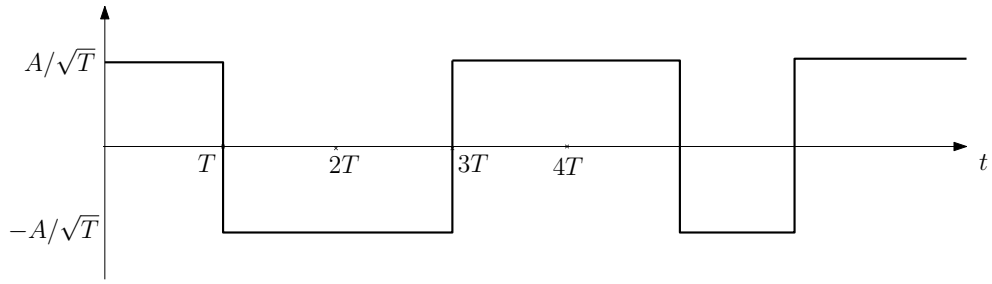
$$\ldots 0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0 \ldots \longrightarrow X_0, X_1, X_2, \ldots \longrightarrow \sum_k X_k p(t - kT)$$

## Rate of Transmission

The modulated baseband signal is $x_b(t) = \sum_k X_k p(t - kT)$.
With the rect pulse shape

$$p(t) = \begin{cases} \frac{1}{\sqrt{T}} & \text{for } t \in (0, T] \\ 0 & \text{otherwise} \end{cases}$$

and $X_k \in \{+A, -A\}$, $x_b(t)$ looks like




Every $T$ seconds, a new symbol is introduced by shifting the pulse and modulating its amplitude with the symbol.

The *transmission rate* is $\frac{1}{T}$ **symbols/sec** or $\frac{\log_2 M}{T}$ **bits/second**



## Desirable Properties of the Pulse Shape $p(t)$

$p(t)$ is chosen to satisfy the following important objectives:

1. We want $p(t)$ to decay quickly in time, i.e., the effect of *symbol* $X_k$ should not start much before $t = kT$ or last much beyond $t = (k+1)T$

2. We want $p(t)$ to be approximately band-limited.
   For a fixed sequence of symbols $\{X_k\}$, the spectrum of $x_b(t)$ is

$$X_b(f) = \mathcal{F}\left[\sum_k X_k p(t - kT)\right] = P(f) \sum_k X_k \, e^{-j2\pi fkT}$$

*Hence the bandwidth of $x_b(t)$ is the same as that of the pulse $p(t)$*

3. The retrieval of the information sequence from the *noisy* received waveform $x_b(t) + n(t)$ should be simple and relatively reliable. In the absence of noise, the symbols $\{X_k\}_{k \in \mathbb{Z}}$ should be recovered perfectly at the receiver.

# Orthonormality of pulse shifts

Consider the third objective, namely, simple and reliable detection.

To achieve this, the pulse is chosen to have the following "**orthonormal shifts**" property:

$$\int_{-\infty}^{\infty} p(t-kT)p(t-mT)\,dt = \begin{cases} 1 & \text{if } k=m \\ 0 & \text{if } k \neq m \end{cases} \tag{1}$$

We'll see how this property makes signal detection at the Rx simple

- This property is satisfied by the rect pulse shape

$$p(t) = \begin{cases} \frac{1}{\sqrt{T}} & \text{for } t \in (0, T] \\ 0 & \text{otherwise} \end{cases}$$

- The sinc pulse $p(t) = \frac{1}{\sqrt{T}} \operatorname{sinc}\left(\frac{\pi t}{T}\right)$ also has orthonormal shifts! (You will show this in Examples Paper 9, Q.2)



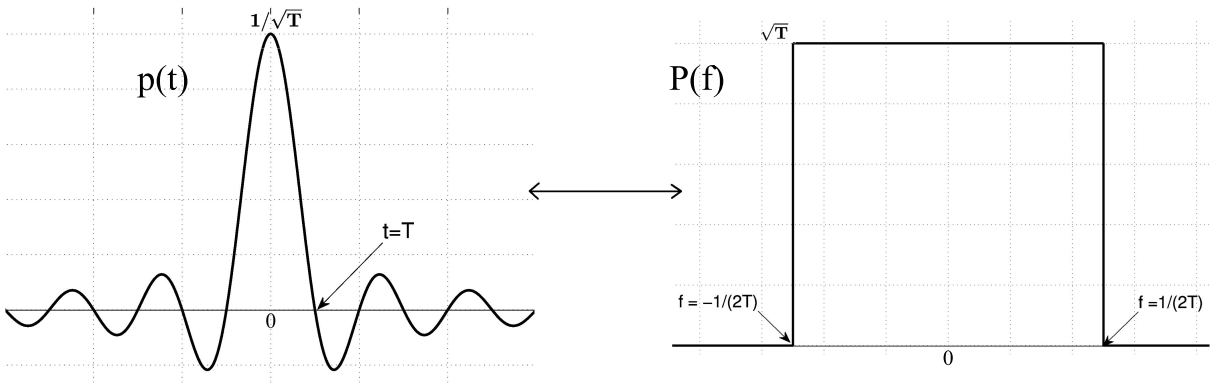
# Time Decay vs. Bandwidth Trade-off

The first two objectives say that we want $p(t)$ to:

1. Decay quickly in time
2. Be approximately band-limited

But ... faster decay in time $\Leftrightarrow$ larger bandwidth

Consider the pulse $p(t) = \frac{1}{\sqrt{T}} \operatorname{sinc}\left(\frac{\pi t}{T}\right)$

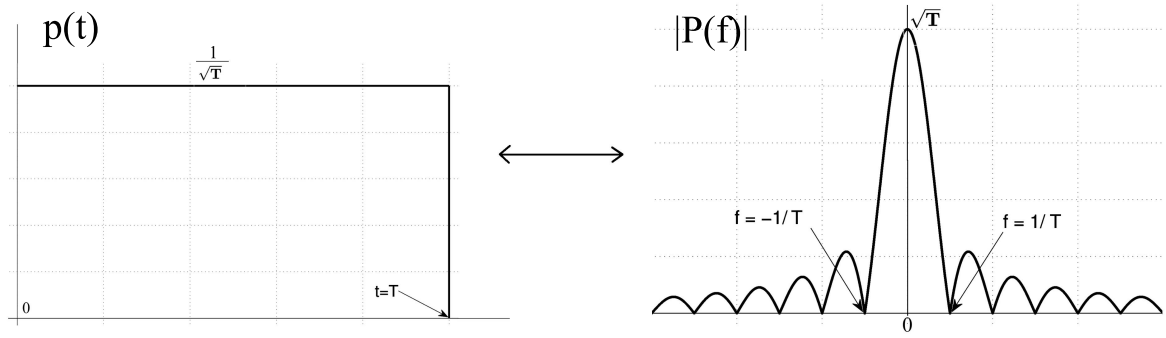



The sinc is perfectly band-limited to $W = \frac{1}{2T}$

But decays slowly in time $|p(t)| \sim \frac{1}{|t|}$

Next consider the rectangular pulse

$$p(t) = \begin{cases} \frac{1}{\sqrt{T}} & \text{for } t \in (0, T] \\ 0 & \text{otherwise} \end{cases}$$

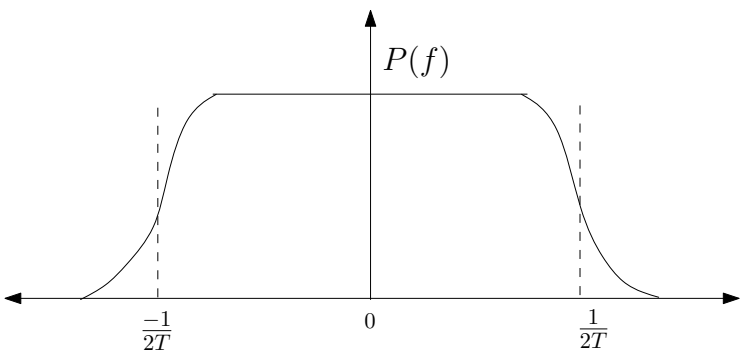



This pulse is perfectly time-limited to the symbol interval $[0, T)$. But ...

- Decays slowly in freq. $|P(f)| \sim \frac{1}{|f|}$
- Main-lobe bandwidth $= \frac{1}{T}$



In practice, the pulse shape is often chosen to have a *raised cosine* spectrum




Bandwidth slightly larger than $\frac{1}{2T}$; decay in time $|p(t)| \sim \frac{1}{|t|^3}$

A happy compromise!

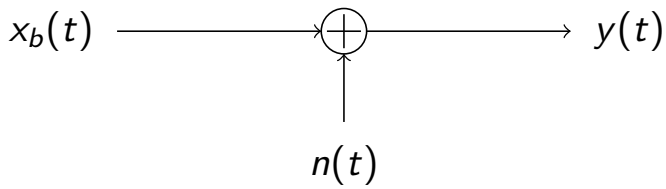
- More on raised cosine pulses in 3F4
- For intuition, it often helps to envision $x_b(t)$ with a rect pulse, though it is never used in practice

# PAM Demodulation

Now, assume that we have picked a constellation and a pulse shape satisfying the objectives, and we transmit the baseband waveform

$$x_b(t) = \sum_k X_k p(t - kT)$$

over a *baseband* channel $y(t) = x_b(t) + n(t)$




How does the receiver recover the information symbols $\{X_0, X_1, X_2, \ldots\}$ from $y(t)$?

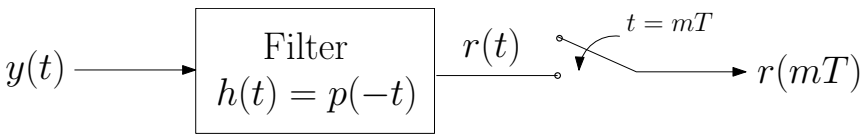
- This process is called *demodulation*
- We will see that the orthonormal shift property of $p(t)$ leads to a simple and elegant demodulator



# Matched Filter Demodulator

Let us first understand the operation assuming no noise, i.e.,

$$y(t) = x_b(t) = \sum_k X_k p(t - kT)$$

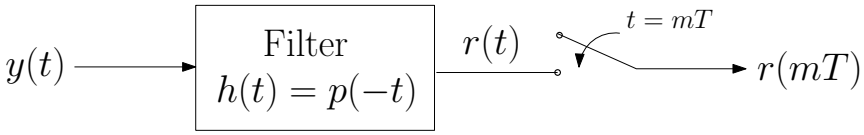



$y(t)$ is passed through a filter with impulse response $h(t) = p(-t)$

This is called a **matched filter**. The filter output is

$$\begin{aligned} r(t) = y(t) \star h(t) = x_b(t) \star h(t) \quad &\text{(assuming no noise)} \\ &= \int_{-\infty}^{\infty} x_b(\tau) h(t - \tau) d\tau \\ &= \int_{-\infty}^{\infty} x_b(\tau) p(\tau - t) d\tau \end{aligned}$$

# The Output of the Matched Filter




$$r(t) = \int_{-\infty}^{\infty} x_b(\tau)p(\tau - t)d\tau = \sum_k X_k \int_{-\infty}^{\infty} p(\tau - kT)p(\tau - t)d\tau$$

By sampling the filter output at time $t = mT$, $m = 0, 1, 2, \ldots$, you get

$$r(mT) = \sum_k X_k \int_{-\infty}^{\infty} p(\tau - kT)p(\tau - mT)d\tau = X_m$$

because of the *orthonormal shifts* property of $p(t)$

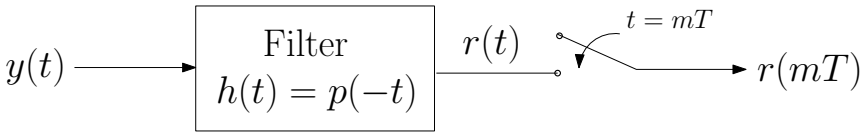
$$\int_{-\infty}^{\infty} p(\tau - kT)p(\tau - mT)d\tau = \begin{cases} 1 & \text{if } k = m \\ 0 & \text{if } k \neq m \end{cases}$$

The *orthonormal shifts* property is crucial for this demodulator to work!



# Demodulation with Noisy $y(t)$

Now consider the noisy case. The receiver gets $y(t) = x(t) + n(t)$




The matched filter output is

$$\begin{aligned} r(t) &= y(t) \star h(t) = x_b(t) \star h(t) \ + \ n(t) \star h(t) \\ &= \sum_k X_k \int_{-\infty}^{\infty} p(\tau - kT)p(\tau - t)d\tau \ + \ \int_{\infty}^{\infty} n(\tau)p(\tau - t)d\tau \end{aligned}$$

Sampling at $t = mT$, $m = 0, 1, 2, \ldots$, we now get

$$r(mT) = X_m + N_m$$

where $N_m$ is noise part of the filter output at time $mT$:

$$N_m = \int_{\infty}^{\infty} n(\tau)p(\tau - mT)d\tau$$

## Properties of the Noise

Let us denote $r(mT)$, the sampled output at time $mT$, by $Y_m$.

$$Y_m = X_m + N_m, \quad m = 0, 1, 2, \ldots$$

Note that this is a *discrete-time channel*. We have converted the continuous-time problem into a discrete-time one of detecting the symbols $X_m$ from the noisy outputs $Y_m$.

- To do this, we first need to understand the properties of the noise $N_m$. Recall that

$$N_m = \int_{\infty}^{\infty} n(\tau)p(\tau - mT)d\tau$$

- $N_m$ is a random variable whose distribution depends on the statistics of the *random process* $n(t)$.

You will learn about random processes and their characterisation in 3F1 & 3F4, but this is outside the scope of this course. For now, we will directly specify the distribution of $N_m$ and analyse the detection problem.



$$Y_m = X_m + N_m, \quad m = 0, 1, 2, \ldots$$

Modelling $n(t)$ as a Gaussian process leads to the following **important** characterisation of $N_m$:

- For each $m$, $N_m$ is a *Gaussian random variable* with zero mean, and variance $\sigma^2$ that can be estimated empirically
- Further $N_1, N_2, \ldots$ are *independent*
- Thus the sequence of random variables $\{N_m\}, m = 0, 1, \ldots$ are independent and identically distributed as $\mathcal{N}(0, \sigma^2)$.

### Detection

- At the Rx, how do we detect the information symbol $X_m$ from $Y_m$ for $m = 0, 1, \ldots$?
- Remember that each $X_m$ belongs to the *symbol constellation*

# Detection for Binary PAM

Let's start with a simple binary constellation, then generalise.

Consider a constellation where each $X_m \in \{-A, A\}$. This is called *binary* PAM or BPSK ('Binary Phase Shift Keying')

$$Y = X + N$$

The detection problem is now:

*Given $Y$, how to decide whether $X = A$ or $X = -A$?*

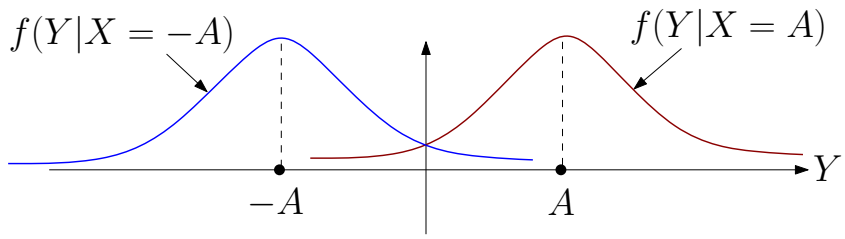
Observe that:

$$Y = A + N \text{ if } X = A \quad \text{and} \quad Y = -A + N \text{ if } X = -A$$

- $N$ is distributed as $\mathcal{N}(0, \sigma^2)$
- Therefore the pdf $f(Y|X = A)$ is Gaussian with mean $A$ and variance $\sigma^2$
- Similarly the pdf $f(Y|X = -A)$ is Gaussian with mean $-A$ and variance $\sigma^2$

Note: Adding a constant to a random variable just shifts the mean, does not change the shape of the distribution



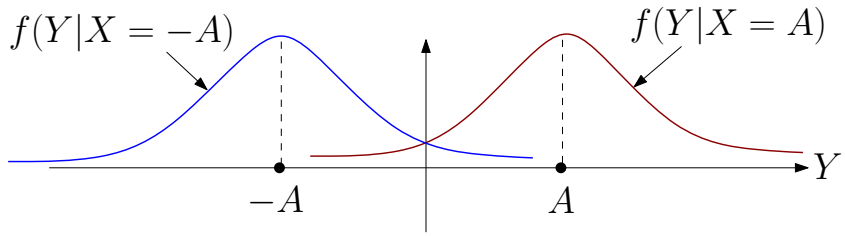



Let $\hat{X}$ denote the decoded symbol. When the symbols $A$ and $-A$ are *a priori* equally likely, the optimal detection rule is:

$$\hat{X} = A \text{ if } f(Y \mid X = A) \geq f(Y \mid X = -A)$$
$$\hat{X} = -A \text{ if } f(Y \mid X = -A) > f(Y \mid X = A)$$

"*Choose the symbol from which $Y$ is most likely to have occurred*"

- This decoder is called the **maximum-likelihood decoder**
- This decoder is intuitive and seems sensible, and is in fact, the optimal detection rule when all the constellation symbols are equally likely (we will not prove this here)
- It is then a special case of the Maximum a Posteriori (MAP) detection rule, which minimises the probability of detection error (module 4F5)




The detection rule can be compactly written as

$$\hat{X} = \underset{x \in \{A,-A\}}{\arg\max}\ f(Y|X=x)$$

$$\begin{aligned}\hat{X} &= \underset{x \in \{A,-A\}}{\arg\max}\ f(Y|X=x) \\ &= \underset{x \in \{A,-A\}}{\arg\max}\ \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(Y-x)^2/2\sigma^2} \quad = \quad \underset{x \in \{A,-A\}}{\arg\min}\ (Y-x)^2\end{aligned}$$

Thus the detection rule is just: $\hat{X} = A$ if $Y \geq 0$, $\hat{X} = -A$ if $Y < 0$

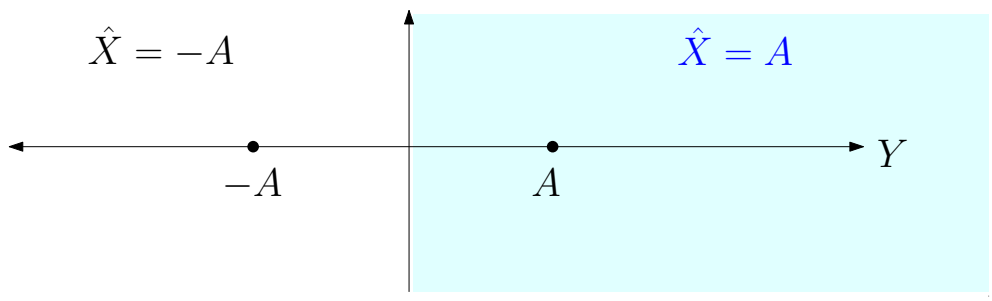
*"Choose the constellation symbol closest to the output $Y$"*



## Decision Regions

The detection rule partitions the space of $Y$ (the real line) into **decision regions**.

For binary PAM, we just derived the following decision regions:




Q: When does the detector make an error?

A: When $X = A$ and $Y < 0$, or When $X = -A$ and $Y > 0$

We will calculate the probability of error shortly, but let's first find the detection rule for general PAM constellations

# Detection for General PAM Constellations

The detection rule can easily be extended to a general constellation $\mathcal{C}$

- E.g., $\mathcal{C}$ may be the 3-ary constellation $\{-2A, 0, 2A\}$ or a 4-ary constellation $\{-3A, -A, A, 3A\}$
- The maximum-likelihood principle is the same: "*Choose the constellation symbol from which $y$ is most likely to have occurred*"

$$\hat{X} = \arg\max_{x \in \mathcal{C}} f(Y|X = x)$$
$$= \arg\max_{x \in \mathcal{C}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(Y-x)^2/2\sigma^2} = \arg\min_{x \in \mathcal{C}} (Y - x)^2$$

Thus, the detection rule for *any* PAM constellation boils down to:
"*Choose the constellation symbol closest to the output $Y$*"



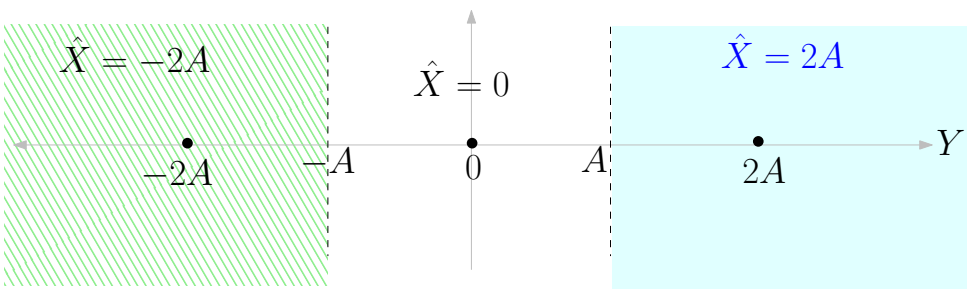
# Example: 3-ary PAM

$$Y = X + N, \quad N \sim \mathcal{N}(0, \sigma^2)$$

What is the optimal detection rule and the associated decision regions if $X$ belongs to the 3-ary constellation $\{-2A, 0, 2A\}$?

The "nearest symbol to $Y$" decoding rule yields

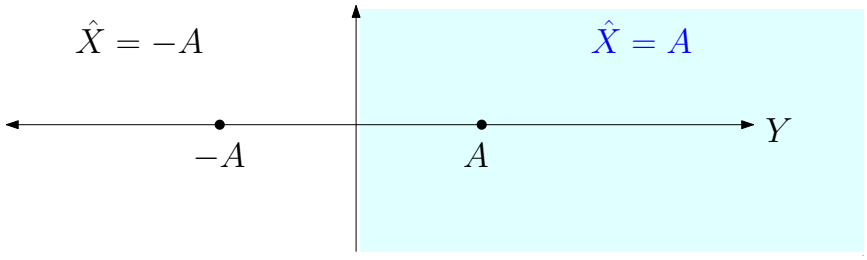
$$\hat{X} = \begin{cases} -2A & \text{if } Y < -A \\ 0 & \text{if } -A \leq Y < A \\ 2A & \text{if } Y > A \end{cases}$$



# Probability of Detection Error

$$Y = X + N$$

Consider binary PAM with $X \in \{A - A\}$. The decision regions are:




The detector makes an error when $X = A$ and $Y < 0$, or when $X = -A$ and $Y > 0$

The probability of detection error is

$$\begin{aligned} P_e &= P(\hat{X} \neq X) \\ &= P(X = -A)P(\hat{X} = A | X = -A) + P(X = A)P(\hat{X} = -A | X = A) \\ &= \tfrac{1}{2}P(\hat{X} = A \mid X = -A) + \tfrac{1}{2}P(\hat{X} = -A \mid X = A) \end{aligned}$$

(The symbols are equally likely $\Rightarrow$ $P(X = A) = P(X = -A) = \frac{1}{2}$



Let us first examine $P(\hat{X} = A \mid X = -A)$

$$\begin{aligned} P(\hat{X} = A \mid X = -A) &= P(Y > 0 \mid X = -A) \\ &= P(-A + N > 0 \mid X = -A) \\ &= P(N > A \mid X = -A) \overset{(a)}{=} P(N > A) \end{aligned}$$

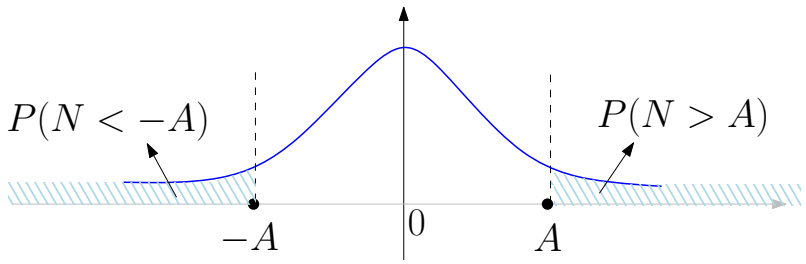
$(a)$ is true because the noise random variable $N$ is **independent** of the transmitted symbol $X$. Similarly,

$$\begin{aligned} P(\hat{X} = -A \mid X = A) &= P(Y < 0 \mid X = A) \\ &= P(A + N < 0 \mid X = A) \\ &= P(N < -A \mid X = A) = P(N < -A) \end{aligned}$$

The probability of detection error is therefore

$$\begin{aligned} P_e &= \tfrac{1}{2}P(\hat{X} = A \mid X = -A) + \tfrac{1}{2}P(\hat{X} = -A \mid X = A) \\ &= \tfrac{1}{2}P(N > A) \;+\; \tfrac{1}{2}P(N < -A) \\ &\overset{(b)}{=} P(N > A) \overset{(c)}{=} P\left(\frac{N}{\sigma} > \frac{A}{\sigma}\right) \end{aligned}$$

- $(b)$ holds due to the symmetry of the Gaussian pdf $\mathcal{N}(0, \sigma^2)$:



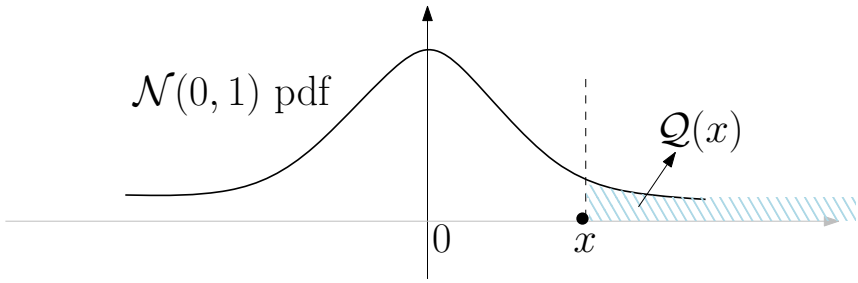

- In $(c)$, we have expressed the probability in terms of a **standard** Gaussian random variable with distribution $\mathcal{N}(0, 1)$
- Recall from 1B Paper 7 (Probability) that if $N$ is distributed as $\mathcal{N}(0, \sigma^2)$ then $\frac{N}{\sigma}$ is distributed as $\mathcal{N}(0, 1)$



## The $\mathcal{Q}$-function

The error probability is usually expressed in terms of the $\mathcal{Q}$-function, which is defined as:

$$\mathcal{Q}(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$




- $\mathcal{Q}(x)$ is the probability that a **standard Gaussian** $\mathcal{N}(0, 1)$ random variable takes value greater than $x$
- Also note that $\mathcal{Q}(x) = 1 - \Phi(x)$, where $\Phi(.)$ is the cdf of a $\mathcal{N}(0, 1)$ random variable

## $P_e$ in terms of the signal-to-noise ratio

The probability of detection error is therefore

$$P_e = P(N > A) = P\left(\frac{N}{\sigma} > \frac{A}{\sigma}\right) = \mathcal{Q}\left(\frac{A}{\sigma}\right) = \mathcal{Q}\left(\sqrt{\frac{E_s}{\sigma^2}}\right)$$

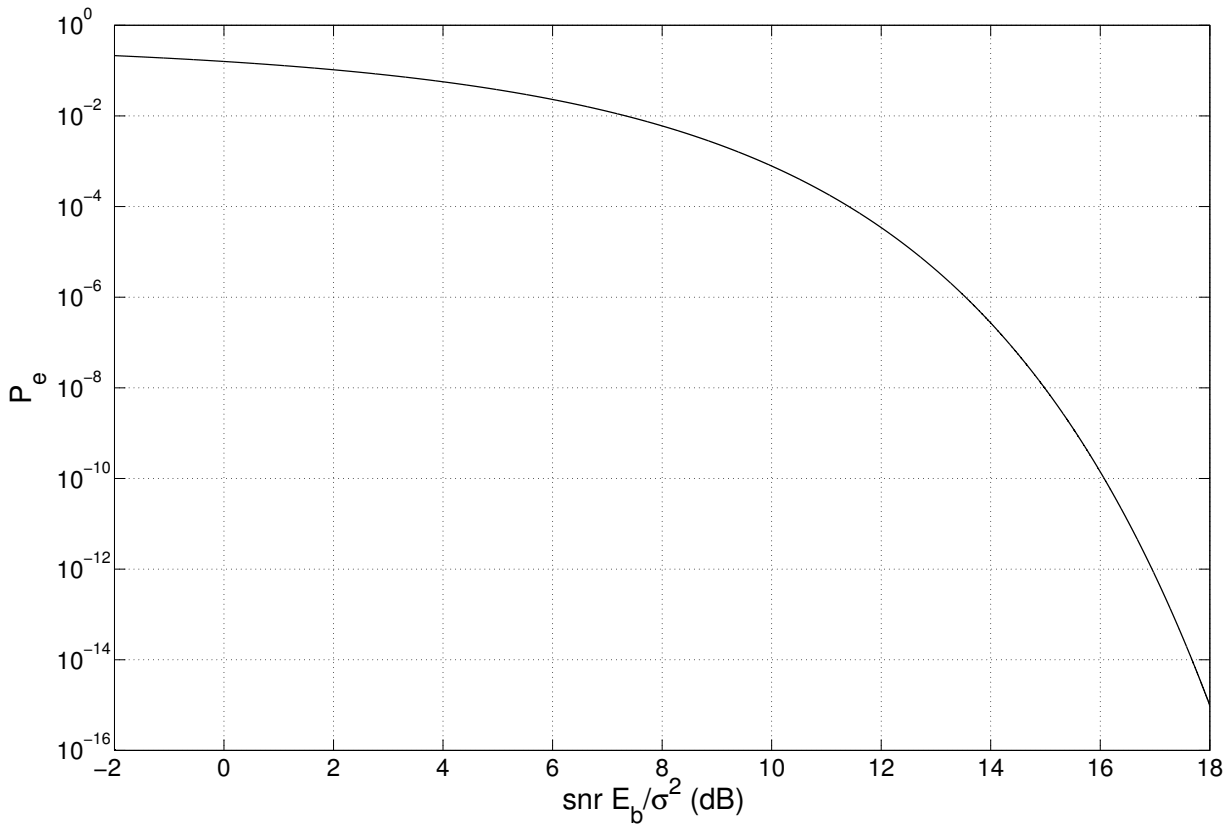
where $E_s$ is the *average energy per symbol* of the constellation:

$$E_s = \frac{1}{2}(A^2 + (-A)^2) = A^2$$

- For a binary constellation, each symbol corresponds to 1 bit. $\Rightarrow$ the *average energy per bit* $E_b$ is also equal to $A^2$ in this case
- For an $M$-point constellation, the average energy per symbol $E_s = E_b \log_2 M$
- $\frac{E_b}{\sigma^2}$ is called the signal-to-noise ratio (snr) of the transmission scheme
- $P_e$ can be plotted as a function of the snr $\frac{E_b}{\sigma^2}$ ...



## $P_e$ vs snr for binary PAM




To get $P_e$ of $10^{-3}$, we need *snr* $E_b/\sigma^2 \approx 9$ dB

# Error Probability vs Transmit Power

The probability of error for binary PAM decays rapidly as $snr \uparrow$:

- $\mathcal{Q}(x) \approx e^{-x^2/2}$ for large $x > 0 \quad \Rightarrow \quad P_e \approx e^{-\text{snr}/2}$

Can we set the snr $\frac{E_b}{\sigma^2}$ to be as high as we want, by increasing $E_b$? (i.e., by increasing $A$ since $E_b = E_s = A^2$ for binary PAM)

- The problem is that transmitted power also increases!
- Intuition: 1 symbol with transmitted every $T$ seconds with average energy $E_s \Rightarrow$ transmit power is $E_s/T$
- Thus as you increase the snr, you battery drains faster!

$$x_b(t) = \sum_k X_k \, p(t - kT)$$

For any PAM constellation the power of the baseband PAM signal $x_b(t)$ is $\frac{E_s}{T} = \frac{E_b \log_2 M}{T}$, where

- $E_s$ is the average symbol energy of the constellation.
- $E_b$ is the average energy per bit

# Power of PAM signal

Consider the PAM signal from time $-nT$ to $nT$, carrying symbols $X_{-n}, X_{-n+1}, \ldots, X_n$ drawn randomly from the constellation

$$P_{PAM} = \lim_{n\to\infty} \frac{1}{2nT} \mathbb{E} \int_{-nT}^{nT} \left( \sum_{k=-n}^{n} X_k \, p(t-kT) \right)^2 dt \qquad (2)$$

where the expectation is needed because the symbols $\{X_k\}$ are *random* symbols drawn uniformly from the PAM constellation.

We write $\left( \sum_{k=-n}^{n} X_k \, p(t-kT) \right)^2$

$$= \sum_{k=-n}^{n} X_k^2 \, p^2(t-kT) + \sum_{k=-n}^{n} \sum_{j\neq k} X_k X_j \, p(t-kT) p(t-jT)$$

Plug back into (2), integrate each of these sums separately ...



$$\text{First term} = \lim_{n\to\infty} \frac{1}{2nT} \sum_{k=-n}^{n} \mathbb{E}[X_k^2] \int_{-nT}^{nT} p^2(t-kT)\, dt$$

$$\overset{(a)}{=} \lim_{n\to\infty} \frac{E_s}{2nT} \sum_{k=-n}^{n} \int_{-nT}^{nT} p^2(t-kT)\, dt$$

$$\overset{(b)}{=} \frac{E_s}{T} \lim_{n\to\infty} \frac{2n+1}{2n} = \frac{E_s}{T}$$

$(a)$ holds because $\mathbb{E}[X_k^2] = E_s$, the average symbol energy. $(b)$ holds because the pulse shape $p(t)$ has unit energy.

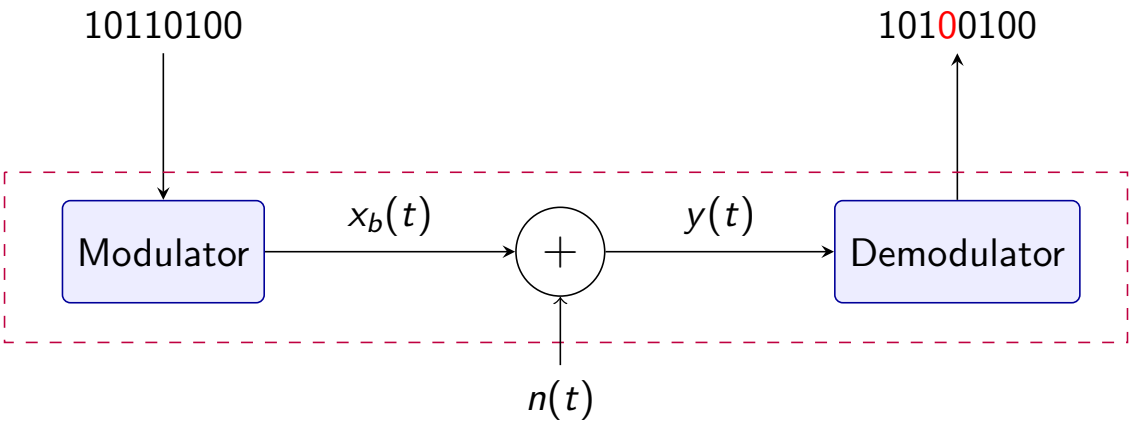
The second term is

$$\lim_{n\to\infty} \frac{1}{nT} \sum_{k=-n}^{n} \sum_{j\neq k} \mathbb{E}[X_k X_j] \int_{-nT}^{nT} p(t-kT) p(t-jT)\, dt = 0$$

because: a) $\mathbb{E}[X_k X_j] = \mathbb{E}[X_k]\mathbb{E}[X_j] = 0$ as the symbols $X_j, X_k$ are independent and the symbol constellation is symmetric around 0; b) further the *orthogonal shifts* property of $p(t)$ implies that the integral also $\to 0$.

Hence $P_{PAM} = E_s/T$. $\square$

# Pulse Amplitude Modulation - The Key Points




PAM is a way to map a sequence of information bits to a *continuous-time baseband* waveform

1. Pick a constellation, map the information bits to symbols $X_1, X_2, \ldots$ in the constellation
2. These symbols then modulate the amplitude of a *pulse shape* $p(t)$ to generate the baseband waveform $x_b(t)$

$$x_b(t) = \sum_k X_k p(t - kT)$$



Desirable properties of the pulse shape $p(t)$:

- $p(t)$ should decay quickly in time; its bandwidth $W$ shouldn't be too large
- *Orthonormal* shifts property for simple and reliable decoding

At the receiver, first **demodulate** then **detect**:

- The demodulator is a *matched filter* with IR $h(t) = p(-t)$
- Matched filter output is *sampled* at times $\ldots, 0, T, 2T, \ldots$. At time $mT$, the output is

$$Y_m = X_m + N_m$$

  $N_m$ is Gaussian noise with zero mean and variance $\sigma^2$ that can be empirically estimated
- Detection rule: $\hat{X}_m =$ the constellation symbol closest to $Y_m$

Probability of detection error can be calculated:

- Decays exponentially with snr $E_s/\sigma^2$
- $E_s$ is average energy/symbol of the constellation; power of PAM waveform $x_b(t)$ is $E_s/T$