Modulation

*Modulation* is the process by which some characteristic of a carrier wave is varied in accordance with an information bearing signal.

A commonly used carrier is a sinusoidal wave, e.g., \( \cos(2\pi f_c t) \). \( f_c \) is called the *carrier frequency*.

- We are allotted a certain bandwidth centred around \( f_c \) for our information signal.
- E.g. BBC Cambridgeshire: \( f_c = 96 \) MHz, information bandwidth \( \approx 200 \) KHz.
- Q: Why is \( f_c \) usually large?
  A: Antenna size \( \propto \lambda_c \) \( \Rightarrow \) larger frequency, smaller antennas!
Analogue vs. Digital Modulation

**Analogue Modulation**: A *continuous information signal* \( x(t) \) (e.g., speech, audio) is used to directly modulate the carrier wave.

We’ll study two kinds of analogue modulation:

1. **Amplitude Modulation** (AM): Information \( x(t) \) modulates the *amplitude* of the carrier wave
2. **Frequency Modulation** (FM): Information \( x(t) \) modulates the *frequency* of the carrier wave

We’ll learn about:
- Power & bandwidth of AM & FM signals
- Tx & Rx design

In the last 4 lectures, we will study *digital* modulation:
- \( x(t) \) is first digitised into bits
- Digital modulation then used to transport bits across the channel

**Amplitude Modulation (AM)**

- Information signal \( x(t) \), carrier \( \cos(2\pi f_c t) \)
- The transmitted AM signal is
  \[
  s_{AM}(t) = [a_0 + x(t)] \cos(2\pi f_c t)
  \]
- \( a_0 \) is a positive constant chosen so that \( \max_t |x(t)| < a_0 \)
- The *modulation index* of the AM signal is defined as
  \[
  m_A = \frac{\max_t |x(t)|}{a_0}
  \]

  “The percentage that the carrier’s amplitude varies above and below its unmodulated level”

*Why is the modulation index important?*

\( m_a < 1 \) is desirable because we can extract the information signal \( x(t) \) from the modulated signal by *envelope detection*. 
When modulation index $> 1$:
- Phase reversals occur
- $x(t)$ cannot be detected by tracing the +ve envelope

AM Receiver - Envelope Detector

- On the positive half-cycle of the input signal, capacitor $C$ charges *rapidly* up to the peak value of input $s_{AM}(t)$
- When input signal falls below this peak, diode becomes reverse-biased: capacitor discharges *slowly* through load resistor $R_L$
- In the next positive half-cycle, when input signal becomes greater than voltage across the capacitor, diode conducts again until next peak value
- Process repeats . . .

Very inexpensive receiver, but envelope detection needs $m_A < 1$. 

\[
\begin{align*}
V_{out}(t) & = s_{AM}(t) \cdot V(t) \\
& = s_{AM}(t) \cdot V_{peak} \\
& = m_A \cdot s(t) \cdot V_{peak}
\end{align*}
\]
Spectrum of AM

Next, let's look at the spectrum of \( s_{AM}(t) = [a_0 + x(t)] \cos(2\pi f_c t) \)

\[
S_{AM}(f) = \mathcal{F}[s_{AM}(t)]
= \mathcal{F} \left[ a_0 + x(t) \left( e^{j2\pi f_c t} + e^{-j2\pi f_c t} \right) / 2 \right]
= \frac{a_0}{2} \left[ \delta(f - f_c) + \delta(f + f_c) \right] + \frac{1}{2} \left[ X(f - f_c) + X(f + f_c) \right]
\]

(\( \mathcal{F}[.] \) denotes the Fourier transform operation)
Example

\[ S_{\text{AM}}(f) = \frac{a_0}{2} [\delta(f - f_c) + \delta(f + f_c)] + \frac{1}{2} [X(f - f_c) + X(f + f_c)] \]

Properties of AM

1. **Bandwidth**: From spectrum calculation, we see that if \( x(t) \) is a baseband signal with (one-sided) bandwidth \( W \), the AM signal \( s_{\text{AM}}(t) \) is passband with bandwidth

   \[ B_{\text{AM}} = 2W \]

2. **Power**: We now prove that the power of the AM signal is

   \[ P_{\text{AM}} = \frac{a_0^2}{2} + \frac{P_X}{2} \]

   where \( P_X \) is the power of \( x(t) \)
Power of AM signal

\[ P_{AM} = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} [a_0 + x(t)]^2 \cos^2(2\pi f_c t) \, dt \]

= \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} [a_0 + x(t)]^2 \frac{1 + \cos(4\pi f_c t)}{2} \, dt

= \frac{a_0^2}{2} + \frac{P_X}{2} + \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} [a_0 + x(t)]^2 \cos(4\pi f_c t) \, dt

We now show that the last the last term is 0.

- \cos(4\pi f_c t) is a high-frequency sinusoid with period \( T_c = \frac{1}{2f_c} \).

- \( g(t) = \frac{(a_0 + x(t))^2}{2} \) is a baseband signal which changes much more slowly than \( \cos(4\pi f_c t) \). Hence, with \( T = nT_c \), we have

\[
\frac{1}{T} \int_{0}^{T} g(t) \cos(4\pi f_c t) \, dt \approx \frac{1}{nT_c} \left( \int_{0}^{T_c} g(0) \cos(4\pi f_c t) \, dt + \int_{T_c}^{2T_c} g(T_c) \cos(4\pi f_c t) \, dt + \right.

\left. + \int_{T_c}^{T} g(t) \cos(4\pi f_c t) \, dt + \int_{nT_c}^{(n+1)T_c} g(T_c) \cos(4\pi f_c t) \, dt \right) = 0
\]

Hence \( P_{AM} = \frac{a_0^2}{2} + \frac{P_X}{2} \).

Double Sideband Suppressed Carrier (DSB-SC)

The power of AM signal is

\[ P_{AM} = \frac{a_0^2}{2} + \frac{P_X}{2} \]

- The presence of \( a_0 \) makes envelope detection possible, but requires extra power of \( \frac{a_0^2}{2} \) corresponding to the carrier

- In DSB-SC, we eliminate the \( a_0 \):

  We transmit only the sidebands, and suppress the carrier

![Diagram of AM signal with sidebands suppressed](image-url)
The transmitted DSB-SC wave is

\[ s_{dsb-sc}(t) = x(t) \cos(2\pi f_c t) \]

How to recover \( x(t) \) at the receiver?
Phase reversals ⇒ cannot use envelope detection

DSB-SC receiver

**DSB-SC Receiver**: Product Modulator + Low-pass filter

1. Multiplying received signal by \( \cos(2\pi f_c t) \) gives

\[ v(t) = x(t) \cos^2(2\pi f_c t) = \frac{x(t)}{2} + \frac{x(t) \cos(4\pi f_c t)}{2} \]

   - low freq.
   - high freq.

2. Low-pass filter eliminates the high-frequency component
   Ideal low-pass filter has \( H(f) = \text{constant for } -W \leq f \leq W \), and zero otherwise
Properties of DSB-SC

\[ s_{dsb-sc}(t) = x(t) \cos(2\pi f_c t) \]
\[ S_{dsb-sc}(f) = \frac{1}{2}(X(f + f_c) + X(f - f_c)) \]

- **Bandwidth** of DSB-SC is \( B_{dsb-sc} = 2W \), same as AM
- **Power** of DSB – SC is \( P_{dsb-sc} = \frac{P_X}{2} \)
  follows from AM power calculation
- DSB-SC requires *less power* than AM as the carrier is not transmitted
- But DSB-SC receiver is *more complex* than AM!
  We assumed that receiver can generate locally generate a frequency \( f_c \) sinusoid that is synchronised perfectly in phase and frequency with transmitter’s carrier
- Effect of phase mismatch at Rx is explored in Examples paper

**Single Sideband Suppressed Carrier (SSB-SC)**

DSB-SC transmits less power than AM. Can we also save bandwidth?

- \( x(t) \) is real \( \Rightarrow X(-f) = X^*(f) \)
  \( \Rightarrow \) Need to specify \( X(f) \) only for \( f > 0 \)
- In other words, transmission of both sidebands is not strictly necessary: we could obtain one sideband from the other!

\[ S_{ssb-sc}(f) \]
\[ X(f) \]
\[ -W \quad 0 \quad W \]
\[ f \]
\[ -f_c - W \quad -f_c \quad f_c \quad f_c + W \]

- **Bandwidth** is \( B_{ssb-sc} = W \), half of that of AM or DSB-SC!
- **Power** is is \( P_{ssb-sc} = \frac{P_X}{4} \), half of DSB-SC
Summary: Amplitude Modulation

You can now do Questions 1–5 on Examples Paper 8.
Frequency Modulation (FM)

In FM, the information signal $x(t)$ modulates the instantaneous frequency of the carrier wave.

The instantaneous frequency $f(t)$ is varied linearly with $x(t)$:

$$f(t) = f_c + k_f x(t)$$

This translates to an instantaneous phase $\theta(t)$ given by

$$\theta(t) = 2\pi \int_0^t f(u) \, du = 2\pi f_c t + 2\pi k_f \int_0^t x(u) \, du$$

The modulated FM signal

$$s_{FM}(t) = A_c \cos(\theta(t)) = A_c \cos\left(2\pi f_c t + 2\pi k_f \int_0^t x(u) \, du\right)$$

- $A_c$ is the carrier amplitude
- $k_f$ is called the frequency-sensitivity factor

Example

What information signal does this FM wave correspond to?

(a) a constant, (b) a ramp, (c) a sinusoid, (d) no clue
FM Demodulation

At the receiver, how do we recover \( x(t) \) from the FM wave? (ignoring effects of noise)

\[
s_{\text{FM}}(t) = A_c \cos \left( 2\pi f_c t + 2\pi k_f \int_0^t x(u) du \right)
\]

The derivative is

\[
\frac{ds_{\text{FM}}(t)}{dt} = -2\pi A_c [f_c + k_f x(t)] \sin \left( 2\pi f_c t + 2\pi k_f \int_0^t x(u) du \right)
\]

- The derivative is a passband signal with amplitude modulation by \([f_c + k_f x(t)]\)
- If \( f_c \) large enough, we can recover \( x(t) \) by envelope detection of \( \frac{ds_{\text{FM}}(t)}{dt} \)!
- Hence FM demodulator is a differentiator + envelope detector
- Differentiator: \( \frac{d}{dt} \xrightarrow{\mathcal{F}} j2\pi f \) (frequency response). See Haykin-Moher book for details on how to build a differentiator

Properties of FM

\[
s_{\text{FM}}(t) = A_c \cos \left( 2\pi f_c t + 2\pi k_f \int_0^t x(u) du \right)
\]

- **Power** of FM signal = \( \frac{A_c^2}{2} \), regardless of \( x(t) \)
- Non-linearity: \( FM(x_1(t) + x_2(t)) \neq FM(x_1(t)) + FM(x_2(t)) \)
- FM is more robust to additive noise than AM.
  Intuitively, this is because the message is “hidden” in the frequency of the signal rather than the amplitude.
- But this robustness comes at the cost of increased transmission bandwidth
- What is the bandwidth of the FM signal \( s_{\text{FM}}(t) \)?
  The spectral analysis is a bit complicated, but we will do it for a simple case . . . where \( x(t) \) is a sinusoid (a pure tone)
FM modulation of a tone
Consider FM modulation of a tone $x(t) = a_x \cos(2\pi f_x t)$. We have

$$f(t) = f_c + k_f a_x \cos(2\pi f_x t)$$
$$\theta(t) = 2\pi f_c t + \frac{k_f a_x}{f_x} \sin(2\pi f_x t)$$

- $\Delta f = k_f a_x$ is called the frequency deviation
  $\Delta f$ is the max. deviation of the carrier frequency $f(t)$ from $f_c$
- $\beta = \frac{k_f a_x}{f_x} = \frac{\Delta f}{f_x}$ is called the modulation index
  $\beta$ is the max. deviation of the carrier phase $\theta(t)$ from $2\pi f_c t$

Then the FM signal becomes

$$s_{FM}(t) = A_c \cos (2\pi f_c t + \beta \sin(2\pi f_x t))$$

The spectrum of the FM signal
We want to understand the frequency spectrum of

$$s_{FM}(t) = A_c \cos (2\pi f_c t + \beta \sin(2\pi f_x t))$$

We can write

$$s_{FM}(t) = \text{Re} \left[ A_c e^{j2\pi f_c t + j\beta \sin(2\pi f_x t)} \right]$$
$$= \text{Re} \left[ \tilde{s}(t) e^{j2\pi f_c t} \right]$$

where

$$\tilde{s}(t) = A_c e^{j\beta \sin(2\pi f_x t)}$$

- $\tilde{s}(t)$ is periodic with period $1/f_x$
- Can express using Fourier series (Fundamental frequency $f_x$)

$$\tilde{s}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi f_x nt}$$
You will show in Examples Paper 8 that the Fourier series coefficients of $\tilde{s}(t)$ are

$$c_n = A_c J_n(\beta)$$

where $J_n(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin u - nu)} \, du$

$J_n(.)$ is called the \textit{nth order Bessel function} of the first kind

Thus

$$\tilde{s}(t) = A_c e^{j\beta \sin(2\pi f_x t)} = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi f_x nt} = A_c \sum_{n} J_n(\beta) e^{j2\pi f_x nt}.$$ 

Therefore

$$s_{FM}(t) = \text{Re} \left[ \tilde{s}(t)e^{j2\pi f_c t} \right] = \text{Re} \left[ \sum_{n=-\infty}^{\infty} c_n e^{j2\pi f_x nt} e^{j2\pi f_c t} \right] = \text{Re} \left[ \sum_{n=-\infty}^{\infty} A_c J_n(\beta) e^{j2\pi (f_c + nf_x) t} \right] = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(2\pi (f_c + nf_x) t)$$

Taking Fourier Transforms, the spectrum of the FM signal is

$$S_{FM}(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta) \left[ \delta(f - f_c - nf_x) + \delta(f + f_c + nf_x) \right]$$
Example

What is the spectrum of the FM signal when $x(t)$ is a pure tone and the modulation index $\beta = 5$?
Example ctd.

The spectrum is

$$S_{FM}(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(5) \left[ \delta(f - f_c - nf_x) + \delta(f + f_c + nf_x) \right]$$

Bandwidth of FM signals

To summarise, when $x(t)$ has only a single frequency $f_x$, the spectrum of the FM wave is rather complicated:

- There is a carrier component at $f_c$, and components located symmetrically on either side of $f_c$ at $f_c \pm f_x$, $f_c \pm 2f_x$, . . .
- The absolute bandwidth is infinite, but . . . the side components at $f_c \pm nf_x$ become negligible for large enough $n$

Carson’s rule for the effective bandwidth of FM signals:

1. The bandwidth of an FM signal generated by modulating a single tone is

$$B_{FM} \approx 2\Delta f + 2f_x = 2\Delta f \left(1 + \frac{1}{\beta}\right)$$

2. For an FM signal generated by modulating a general signal $x(t)$ with bandwidth $W$, the bandwidth $B_{FM} \approx 2\Delta f + 2W$

(Recall: for any FM wave, $\Delta f$ is the frequency deviation around $f_c$)
Example

BBC Radio Cambridgeshire: $f_c = 96$ MHz and $\Delta f = 75$ kHz. Assuming that the voice/music signals have $W = 15$ kHz, we have

$$\beta = \frac{\Delta f}{W} = \frac{75}{15} = 5$$

and the bandwidth

$$B_{FM} = 2(\Delta f + W) = 2(75 + 15) = 180 \text{ kHz},$$

while

$$B_{AM} = 2W = 30 \text{ kHz}$$

FM signals have larger bandwidth than AM, but have better robustness against noise.

Summary: Analogue Modulation

Amplitude Modulation with information signal of bandwidth $W$

- **AM** modulated signal: Bandwidth $2W$, high power, simple Rx using envelope detection
- **DSB-SC**: Bandwidth $2W$, lower power, more complex Rx
- **SSB-SC**: Bandwidth $W$, even lower power, Rx similar to DSB-SC

Frequency Modulation with information signal of bandwidth $W$:

- FM signal has constant carrier amplitude $\Rightarrow$ constant power
- Bandwidth of FM signal depends on both $\beta$ and $W$
  Can be significantly greater than $2W$
- Better robustness to noise than AM – the information is “hidden” in the phase