4F7 Spectrum Estimation
Power Spectrum Estimation

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Modification history:
$\leq$ 2008: S. Godsill, P. Rayner
Revised by S.S. Singh

1 Discrete Signals

Widowing effects are observed for continuous time and discrete time signals

Consider the discrete case shown in figure 1.

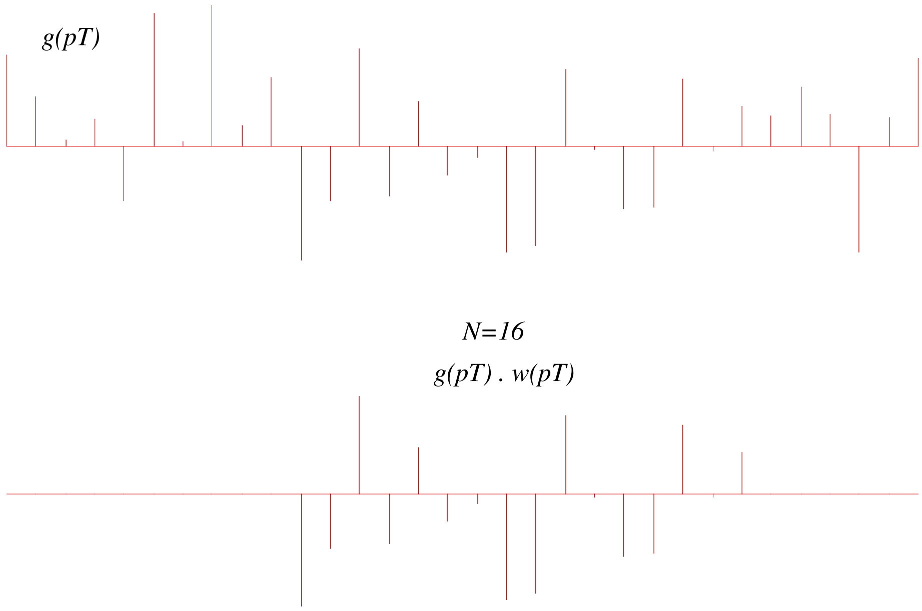

Figure 1: Windowed discrete signal

The sampled values of the window signal are $w_p = w(pT)$ and $g_p = g(pT)$, respectively.

The DTFT of the windowed signal $w_p g_p$ is

$$G(e^{j\omega}) = \sum_{p=-\infty}^{\infty} g_p \, e^{-jp\omega}$$

$$\begin{aligned} G_w(e^{j\omega}) &= \sum_{p=-\infty}^{\infty} g_p \, w_p e^{-jp\omega} \\ &= \sum_{p=-\infty}^{\infty} g_p \left\{ \frac{1}{2\pi} \int_0^{2\pi} W(e^{j\theta}) e^{jp\theta} d\theta \right\} e^{-jp\omega} \\ &= \frac{1}{2\pi} \int_0^{2\pi} W(e^{j\theta}) \sum_{p=-\infty}^{\infty} g_p \, e^{-jp(\omega-\theta)} \, d\theta \end{aligned}$$

$$G_w(e^{j\omega}) = \frac{1}{2\pi} \int_0^{2\pi} W(e^{j\theta}) \, G(e^{j(\omega-\theta)}) \, d\theta$$

Like the continuous case, the spectrum of the windowed signal is the convolution of the infinite duration signal spectrum and the window spectrum.

Note that all discrete time spectra are periodic functions of frequency ω.

As for the continuous case we can consider the use of tapered windows and one class of window functions is the generalised Hamming window given by

$$w_n = \alpha - (1-\alpha)\cos\left(\frac{2\pi}{N}n\right) \text{ for } n = 0 \text{ to } N-1$$

$\alpha = 1$	Rectangular window
$\alpha = 0.5$	Hanning window (Raised Cosine or Cosine Arch)
$\alpha = 0.54$	Hamming window

We can evaluate the spectrum (DTFT) of the generalised window as follows

$$W(e^{j\omega}) = \sum_{p=0}^{N-1} \left\{ \alpha - \frac{(1-\alpha)}{2} \left[e^{j\frac{2\pi}{N}p} + e^{-j\frac{2\pi}{N}p} \right] \right\} e^{-jp\omega}$$

which gives

$$W(e^{j\omega}) = e^{-j(N-1)\frac{\omega}{2}} \left\{ \alpha \frac{\sin(N\frac{\omega}{2})}{\sin(\frac{\omega}{2})} \right.$$

$$\left. + \frac{1-\alpha}{2} \left[e^{-j\frac{\pi}{N}} \frac{\sin\left[\frac{N}{2}(\omega - \frac{2\pi}{N})\right]}{\sin\left[\frac{1}{2}(\omega - \frac{2\pi}{N})\right]} + e^{j\frac{\pi}{N}} \frac{\sin\left[\frac{N}{2}(\omega + \frac{2\pi}{N})\right]}{\sin\left[\frac{1}{2}(\omega + \frac{2\pi}{N})\right]} \right. \right\}$$

This is shown in figure 2 for the Hanning window ($\alpha = 0.5$) and for other values of α.

Many other windows with different side-lobe and central lobe properties are available, e.g. Blackman, Bartlett, Chebyshev, Kaiser, ...

These are all available as Matlab functions, so you can easily display them and their DFT within Matlab.

Matlab demo: `disc_wind.m` (Type window at the Matlab command line for an interactive window display program.) `wvtool` (Built in Matlab window visualization tool.)

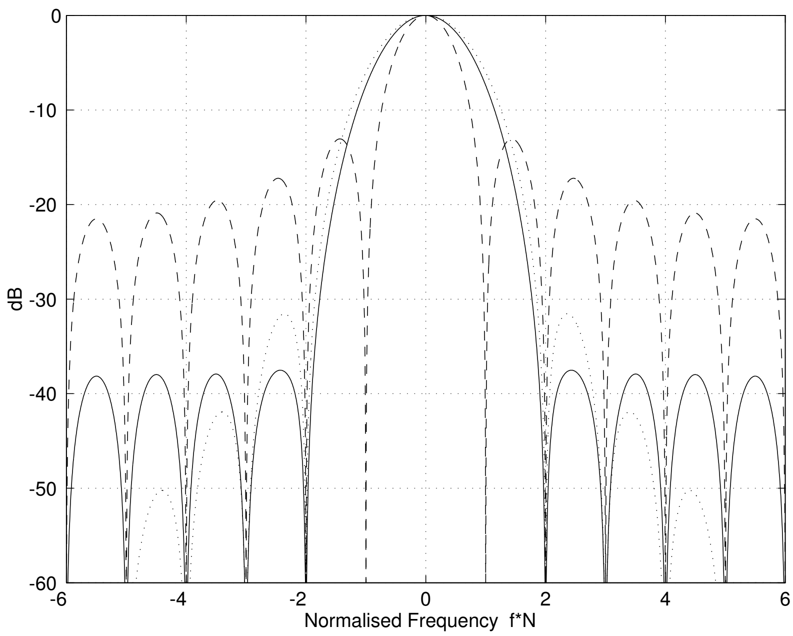

Figure 2: Discrete Window spectra for $\alpha = 1.0$ ($- - -$), $\alpha = 0.5$ ($\cdots$), $\alpha = 0.54$ (–).

2 Power Spectrum Estimation

- We will consider the problem of estimating the Power Spectrum (power spectral density) of a wide-sense stationary random process
- The Fourier transform is an important tool for analysing deterministic signals and is equally important for random processes too
- For a random process (or random signal), it is meaningless to evaluate the Fourier transform of a particular waveform since the random process is an ensemble of discrete time signals
- It is possible to obtain a frequency domain representation by "expressing the Fourier transform in terms of an ensemble average"
- The power spectrum is the Fourier transform of the autocorrelation sequence
- The power spectrum tells us about the expected or average power of a signal at each frequency in the spectrum

- First revise 3F1/3F3 concepts of random processes and power spectrum in continuous and discrete time

Recommended textbooks:

Discrete random signals and statistical signal processing - Charles W. Therrien. Prentice-Hall. Chapter 10

Statistical digital signal processing and modeling - Monson H. Hayes (1996), Wiley: New York, Chapters 3, 8

3 Discrete time random processes-revision

A discrete-time random process (or time series) $\{X_n\}$ can be thought of as a continuous-time random process $\{X(t)\}$ evaluated at times $t = nT$. Four important statistics are the mean, variance, autocovariance and autocorrelation

- The mean of a random process $\{X_n\}$ is $E[X_n]$ and the variance is

$$E[X_n^2] - E[X_n]^2$$

- The autocovariance is

$$E\left\{(X_{n_1} - E[X_{n_1}])\,(X_{n_2} - E[X_{n_2}])\right\}$$

- The autocorrelation is

$$\boxed{R_{XX}[n_1, n_2] = E[X_{n_1} X_{n_2}]}$$

- A process $\{X_n\}$ is wide-sense stationary (WSS) process if i) its mean is a constant, $E[X_n] = c$

for all n, ii) $R_{XX}(n_1, n_2)$ depends only on the difference $k = n_2 - n_1$, i.e.

$$\boxed{R_{XX}[k] = E[X_n X_{n+k}]}, \tag{1}$$

and, iii) the variance is $E[X_n^2] - E[X_n]^2$ finite

- The Power Spectrum or Spectral Density of a WSS random process $\{X_n\}$ is defined as the Discrete time Fourier Transform (DTFT) of $R_{XX}[k]$,

$$\boxed{S_X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} R_{XX}[k]\, e^{-jk\omega}} \tag{2}$$

where T is the sampling period.

- $R_{XX}[k]$ can be recovered from the power spectrum

$$R_{XX}[k] = \frac{1}{2\pi} \int_0^{2\pi} S_X(e^{j\theta}) e^{jk\theta} d\theta$$

- The power spectrum is real, non-negative, even and periodic (as a function of ω)

- For an ergodicrandom process we can estimate expectations by performing time-averaging on a single sample function (or realisation of the porcess), e.g.

$$\mu = E[X_n] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{+N} x_n \ \text{(Mean ergodic)}$$

$$\begin{aligned} & R_{XX}[k] \\ & = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{+N} x_n x_{n+k} \ \text{(Correlation ergodic)} \end{aligned} \tag{3}$$

- Unless otherwise stated, assume that the signals we encounter are both wide-sense stationary and ergodic

- The total power of the signal is

$$R_{XX}[0] = \frac{1}{2\pi} \int_0^{2\pi} S_X(e^{j\theta}) d\theta$$

Since integrating gives the total power, $S_X(e^{j\theta})$, as a function of θ, is a (power spectrum) density

- You can prove the following result (Hayes, pg 99)

$$S_X(e^{j\omega}) = \lim_{N\to\infty} \frac{1}{2N+1} E\left\{ \left| \sum_{n=-N}^{N} x_n e^{-jn\omega} \right|^2 \right\}. \quad (4)$$

  Interpretation: truncate the sequence, compute the Fourier transform, square the absolute value and then compute the expectation.

- Another physical interpretation (Haykin, pg 120): if you filter the random process through an ideal bandpass filter with upper and lower cut off frequencies ω_u and ω_l respectively,

$$H(e^{j\omega}) = 1 \text{ only if } 0 < \omega_l \leq |\omega| \leq \omega_u < \pi)$$

  the power of the output is

$$\frac{1}{\pi}\int_{\omega_l}^{\omega_u} S_X(e^{j\theta})d\theta \approx \frac{1}{\pi}(\omega_u - \omega_l)S_X(e^{j\omega_c})$$

  where $\omega_c = (\omega_u + \omega_l)/2$. Recall the result

$$S_Y(e^{j\omega}) = \left|H(e^{j\omega})\right|^2 S_X(e^{j\omega}).$$

Question: Why is it useful to evaluate power spectrum rather than just take Fourier transforms of single sample functions?

Example: white noise

- Figure 3 shows 3 random realizations from a white noise process and corresponding spectra
- Note that the individual spectra tell us nothing particularly useful because of the randomness between realizations
- Figure 4 shows the mean of 500 random spectra of white noise, i.e. approximating equation 4

We see that the expected flat power spectrum is now well estimated.

- Similar considerations apply to spectrum estimation for non-white random processes
- In practice we only have access to one or a few random realizations from the process. We

will thus use the results for ergodic signals to estimate the required ensemble expectation

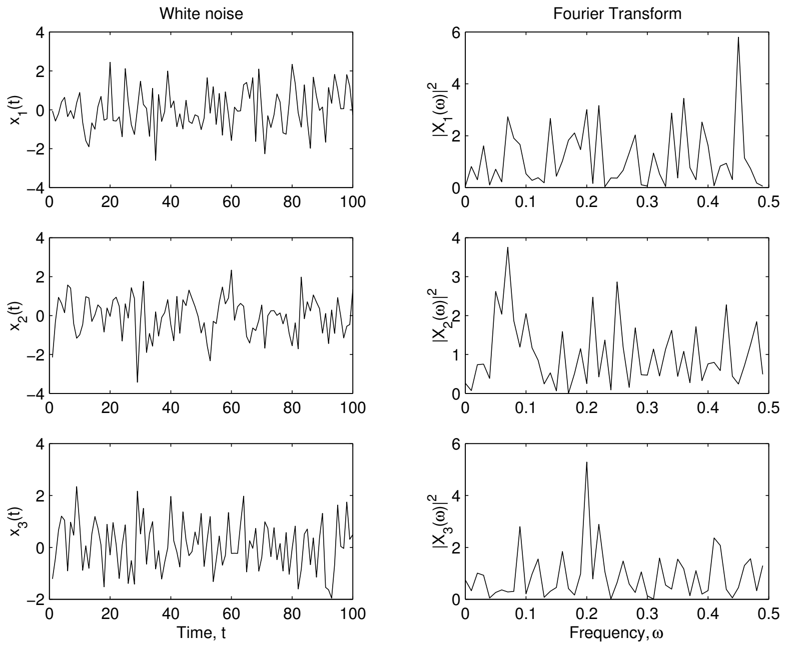

Figure 3: White noise sequences and their Fourier Transforms

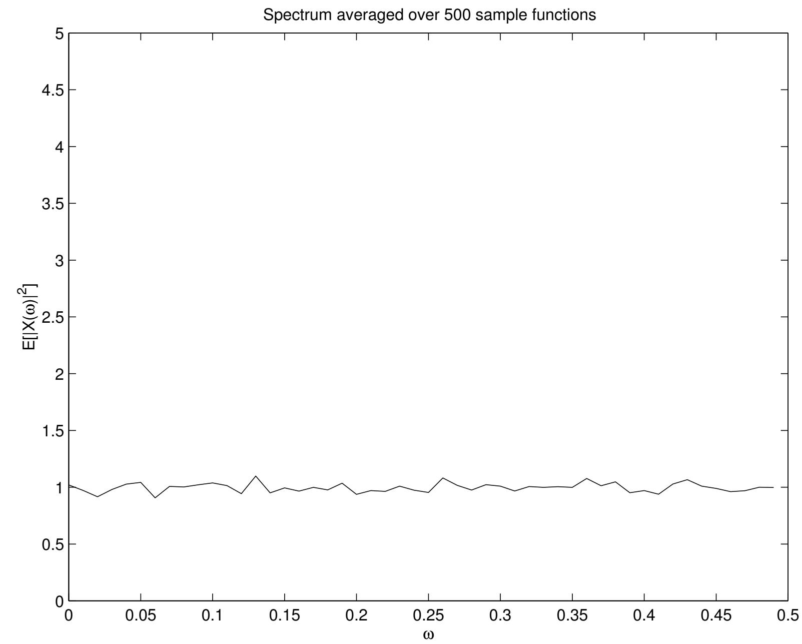

Figure 4: Average of 500 realizations of white noise Fourier Transforms

4 Practical Power Spectrum Estimation

- For an autocorellation ergodic process

$$R_{XX}[k] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x_n x_{n+k}$$

- Given an infinitely long realisation, estimating the power spectrum is straightforward

- In practise we only have a limited dataset, e.g. N data points, $\{x_n\}_{n=0}^{N-1}$

- The second difficulty is that the data may be corrupted by noise or an intefering signal

- Power spectrum estimation techniques must cope with these constraints

- The basic principle is, generally, to estimate the autocorrelation function R_{XX} and then compute the Fourier transform. This gives rise to the Correlogram and Periodogram estimates

- Further improvements can be made if we perform various types of smoothing or averaging - Bartlett, Blackman-Tukey, Welch methods

5 Correlogram and Periodogram Estimates

- These classical techniques are based on the principle of obtaining estimates of the autocorrelation function R_{XX} of the random process and then taking the Fourier transform as in equation 2:

$$S_X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} R_{XX}[k]\, e^{-jk\omega}$$

- If the process is WSS and ergodic, we can estimate R_{XX} based upon equation 3 assuming a correlation ergodic signal:

$$R_{XX}[k] \approx \frac{1}{2N+1} \sum_{-N}^{+N} x_n x_{n+k}$$

- There are several ways to proceed when the number of data points is finite; we consider the consequences of two of these.

Assume that N data points are available, $\{x_n\}_{n=0}^{N-1}$. Two possible estimates of the autocorrelation function are:

1. Sample autocorrelation function (biased estimate):

$$\boxed{\hat{R}_{XX}[k] = \frac{1}{N} \sum_{n=0}^{N-1-k} x_n \, x_{n+k} \qquad 0 \leq k < N} \tag{5}$$

2. Sample autocorrelation function (unbiased estimate):

$$\boxed{\hat{R}_{XX}[k] = \frac{1}{N-k} \sum_{n=0}^{N-1-k} x_n \, x_{n+k} \qquad 0 \leq k < N} \tag{6}$$

- Limits of the sum ensure only available samples are used in the estimators
- (5) is biased since we divide the summation by N rather than $N-k$, the number of terms in the summation.

- Note that the autocorrelation is an even function so that estimates for negative k are given by:
$$\hat{R}_{XX}[-k] = \hat{R}_{XX}[k]$$

- Assume $R_{XX}[k] = 0$ for $|k| > L$, where L is some chosen constant, typically with $L << N$

- The Correlogram estimate for the power spectrum is obtained by taking the DTFT of the sample autocorrelation function, $\hat{R}_{XX}[k]$:

$$\boxed{\hat{S}_X(e^{j\omega}) = \sum_{k=-L}^{L} \hat{R}_{XX}[k]\, e^{-jk\omega}, \qquad L < N}$$

- If the maximum correlation lag is taken to be:

$$L = N - 1$$

then the resulting estimate is:

$$\hat{S}_X(e^{j\omega}) = \sum_{k=-(N-1)}^{N-1} \hat{R}_{XX}[k]\, e^{-jk\omega} \tag{7}$$

- When the biased estimator (5) is used for $\hat{R}_{XX}$, this can be rewritten in terms of the

DTFT of $\{x_0, x_1, ..., x_{N-1}\}$:

$$\boxed{\begin{aligned} \hat{S}_X(e^{j\omega}) &= \frac{1}{N}|X_w(e^{j\omega})|^2 \\ X_w(e^{j\omega}) &= \sum_{n=0}^{N-1} x_n \, e^{-jn\omega} \end{aligned}} \tag{8}$$

which is known as the Periodogram.

- To prove (8), rewrite the biased estimate as:

$$\hat{R}_{XX}[k] = \frac{1}{N}\sum_{n=0}^{N-1-k} x_n\, x_{n+k}$$
$$= \frac{1}{N}\sum_{n=-\infty}^{\infty} v_n\, v_{n+k}$$

where $v_n = w_n x_n$ is a version of x_n truncated by multiplication with a rectangular window:

$$w_n = \begin{cases} 1, & n = 0, 1, ..., N-1 \\ 0, & \text{otherwise} \end{cases}$$

Let $n' = -n$ and define the sequence $u_n = v_{-n}$, $n = 0, \pm 1, \pm 2, \ldots$

$$\hat{R}_{XX}[k] = \frac{1}{N}\sum_{n'=-\infty}^{\infty} v_{-n'}\, v_{(-n'+k)}$$
$$= \frac{1}{N}\sum_{n'=-\infty}^{\infty} u_{n'}\, v_{k-n'} = \frac{1}{N}\{u * v\}(k)$$

i.e. a standard discrete time convolution of $\{u_n\}_n$ with $\{v_n\}_n$.

- Taking the DTFT of both sides we get (by the discrete time convolution theorem):

$$\hat{S}_X(e^{j\omega}) = \frac{1}{N} U(e^{j\omega})\, V(e^{j\omega})$$

where:

$$\begin{aligned} V(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} v_n e^{-jn\omega} = \sum_{n=0}^{(N-1)} x_n e^{-jn\omega} \\ &= X_w(e^{j\omega}) \end{aligned}$$

- and

$$\begin{aligned} U(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} u_n e^{-jn\omega} = \sum_{n=-\infty}^{+\infty} v_{-n} e^{-jn\omega} \\ &= \sum_{n=-N+1}^{0} x_{-n} e^{-jn\omega} = \sum_{n=0}^{N-1} x_n e^{jn\omega} \\ &= X_w^*(e^{j\omega}) \end{aligned}$$

where X_w is the DTFT of the windowed signal $x_n\, w_n$.

- Hence

$$\begin{aligned}\hat{S}_X(e^{j\omega}) &= \frac{1}{N} U(e^{j\omega})\, V(e^{j\omega}) \\ &= \frac{1}{N} X_w^*(e^{j\omega}) X_w(e^{j\omega}) = \frac{1}{N} |X_w(e^{j\omega})|^2\end{aligned}$$