4F7 (Adaptive Filters and) Spectrum Estimation

Fitting the Moving Average Model

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1 Parametric methods: The MA Model $(P = 0)$

- The MA model is a FIR filter driven by white noise. The Yule-Walker equations simplifies to

$$\begin{bmatrix} R_{XX}[0] \\ R_{XX}[1] \\ \vdots \\ R_{XX}[Q] \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_Q \end{bmatrix} \tag{1}$$

- However the solution of this equation is not trivial since the c_i, given in a previous lecture, is the convolution of the MA coefficients b_i and the impulse response of the ARMA model

- The ARMA model impulse response is

$$x_n = \sum_{m=-\infty}^{\infty} h_m \, w_{n-m} \stackrel{\text{(causal)}}{=} \sum_{m=0}^{\infty} h_m \, w_{n-m}$$

- Comparing with the MA model, $\sum_{m=0}^{Q} b_m \, w_{n-m}$, we see that $h_i = b_i$

- Using $h_i = b_i$, the expression for c_r given before may be rewritten as:

$$c_r = \begin{cases} \sum_{q=r}^{Q} b_q \, b_{q-r} & \text{if } r \leq Q \\ 0 & \text{if } r > Q \end{cases} \tag{2}$$

- (2) is valid for negative r too and for $r < 0$

$$c_r = c_{|r|}$$

- The convolution of the following two infinite sequences

$$\ldots\ 0,\ \ b_0,\ b_1,\ \ldots\ b_Q,\ 0,\ \ldots$$
$$\ldots\ 0,\ b_Q,\ \ldots\ b_1,\ b_0,\ 0,\ \ \ldots$$

gives c_r, i.e. $c_r = \left(\{b_{-n}\} * \{b_n\}\right)(r)$

- Let $\mathcal{Z}(\{x_n\}) = \sum_{n=-\infty}^{+\infty} x_n z^{-n}$ be the 'bilateral' z-transform of the sequence $\{x_n\}$

$$\sum_{r=-Q}^{Q} c_{|r|}\, z^{-r} = \mathcal{Z}(\{b_n\} * \{b_{-n}\}) \tag{3}$$

$$= B(z)B(z^{-1}) \quad (\text{since } \mathcal{Z}\{b_{-n}\} = B(z^{-1})) \tag{4}$$

and substituting for c_r from equation 1:

$$\boxed{B(z)\, B(z^{-1}) = \sum_{r=-Q}^{Q} R_{XX}[|r|]\, z^{-r}} \tag{5}$$

- Let the zeros of $B(z)$, $\{z \in \mathbb{C} : B(z) = 0\}$, be $n_1, n_2, \ldots, n_Q$. Then, it is obvious that $n_1^{-1}, n_2^{-1}, \ldots, n_Q^{-1}$ are the zeros of $B(z^{-1})$.

- The zeros of $B(z)B(z^{-1})$ are $\{n_i, n_i^{-1}\}_{i=1}^{Q}$
If a zero n_i lies inside (or on) the unit circle, then the corresponding

inverse zero $1/n_i$ lies outside (or on) the unit circle

- Technical condition: assume all the zeros of $B(z)$ lie inside the unit circle so that the MA process is invertible. Invertible means you can express w_n using x_n and its past values.
- Now identify the zeros of $B(z)$ by finding the zeros of the RHS of (5) that lie within the unit circle.

- Once we have the roots n_i of $B(z)$ it is straightforward to reassemble $B(z)$ from the zeros, up to an unknown scale factor g
- You can always write $B(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_Q z^{-Q}$ as

$$\begin{aligned} B(z) &= g \prod_{i=1}^{Q} (1 - z^{-1} n_i) \\ &= g(b'_0 + b'_1 z^{-1} + b'_2 z^{-2} + \cdots + b'_Q z^{-Q}) \end{aligned}$$

  with $b'_0 = 1$.
- Solve for the scale factor g

$$\underbrace{\sum_{n=0}^{Q} b_n^2 = c_0}_{\text{eqn (2)}} \quad \text{and} \quad \overbrace{c_0 = R_{XX}[0]}^{\text{eqn (1)}}$$

- Hence

$$\sum_{i=0}^{Q} (g b'_i)^2 = R_{XX}[0]$$

from which:

$$g = \sqrt{\frac{R_{XX}[0]}{\sum_{i=0}^{Q} (b'_i)^2}}$$

and finally:

$$b_i = g \times b'_i = \sqrt{\frac{R_{XX}[0]}{\sum_{i=0}^{Q} (b'_i)^2}} b'_i$$

Example It is required to fit an MA model to the correlation data:

$$\begin{bmatrix} R_{XX}[0] \\ R_{XX}[1] \\ R_{XX}[2] \end{bmatrix} = \begin{bmatrix} 4.06 \\ -2.85 \\ .9 \end{bmatrix}$$

Therefore

$$\sum_{r=-Q}^{Q} R_{XX}(r)\, z^{-r}$$

$$= 0.9\, z^{-2} - 2.85\, z^{-1} + 4.06 - 2.85\, z + 0.9\, z^{2}$$

Factorisation of this polynomial gives the roots:

$$0.8333 \pm j\, 0.6455$$

$$0.75 \pm j\, 0.5808 \quad (n_1, n_2 \text{ roots inside unit circle})$$

which are plotted in figure 1. The roots inside the unit circle are identified with $B(z)$.

$$\begin{aligned} B(z) &= g(1 - z^{-1}n_1)(1 - z^{-1}n_2) \\ &= g(1 - z^{-1}(n_1 + n_2) + z^{-2}n_1n_2) \\ &= g(1 - z^{-1}1.5 + z^{-2}0.9) \end{aligned}$$

Thus

$$\begin{aligned} B(z) &= \sqrt{\frac{4.06}{1 + 1.5^2 + 0.9^2}}(1 - 1.5\,z^{-1} + 0.9\,z^{-2}) \\ &= 1 - 1.5\,z^{-1} + 0.9\,z^{-2} \end{aligned}$$

and the corresponding MA model is:

$$x_n = w_n - 1.5\,w_{n-1} + 0.9\,w_{n-2}$$

where w_n is white noise with variance equal to 1.

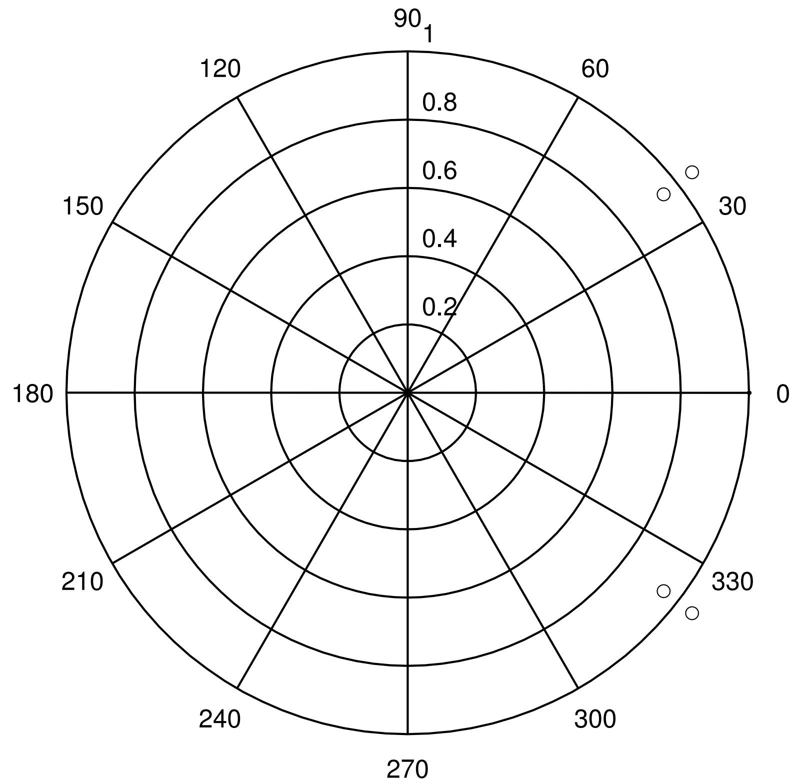

Figure 1: Zeros of $B(z)$, $B(z^{-1})$