Engineering Tripos Part IIB FOURTH YEAR

4F7 Adaptive filters and Spectrum estimation

Examples Paper 2

1. The LMS algorithm is

$$\mathbf{h}(n+1) = \mathbf{h}(n) - \frac{\mu}{2}\nabla J(\mathbf{h})|_{\mathbf{h}=\mathbf{h}(n)}$$

Now compute the partial derivatives.

Compute the expectation:

$$E\{\mathbf{h}(n+1)\} = (1-\mu\alpha)E\{\mathbf{h}(n)\} + \mu E(\mathbf{u}(n)e(n))$$

Use the approximation

$$E\left(\mathbf{u}(n)\mathbf{u}^{\mathrm{T}}(n)\mathbf{h}(n)\right) \approx E\left(\mathbf{u}(n)\mathbf{u}^{\mathrm{T}}(n)\right)E(\mathbf{h}(n))$$

which was verified in lectures for a *block-type* update scheme. Thus

$$E\{\mathbf{h}(n+1)\} = (1-\mu\alpha)E\{\mathbf{h}(n)\} + \mu\mathbf{p} - \mu\mathbf{R}E\{\mathbf{h}(n)\}$$

Replace left and right-hand side by the limit $\overline{\mathbf{h}}$ to get

$$\alpha\overline{\mathbf{h}} = \mathbf{p} - \mathbf{R}\overline{\mathbf{h}}$$
$$\overline{\mathbf{h}} = (\mathbf{R}+\alpha\mathbf{I})^{-1}\mathbf{p}.$$

We denote $\lambda_{\min}$ and $\lambda_{\max}$ the smallest and largest eigenvalues of $\mathbf{R}$. The smallest and largest eigenvalues of $\mathbf{R}+\alpha\mathbf{I}$ are thus equal to $\lambda_{\min}+\alpha$ and $\lambda_{\max}+\alpha$. To ensure convergence,

$$\mu < \frac{2}{\lambda_{\max}+\alpha}.$$

This algorithm can be beneficial if $\lambda_{\min}$ is very small. In this case, the ratio $\lambda_{\max}/\lambda_{\min}$ is large and the speed of convergence is slow. By adding α, it speeds up the convergence of the algorithm since it reduces the eigenvalue spread.

2. Take the expectation of $\widehat{C}$

$$E\left\{\widehat{C}\right\} = a_1E\{y_1\} + a_2E\{y_2\} = (a_1+a_2)C$$

where the results follows since

$$E\{y_1\} = C + E\{e_1\} = C,$$
$$E\{y_2\} = C + E\{e_2\} = C.$$

For the estimate to be unbiased, we require

$$a_1 + a_2 = 1.$$

Compute the variance of the estimate:

$$\begin{aligned} var\left\{\widehat{C}\right\} &= E\left\{\left(\widehat{C} - E\left\{\widehat{C}\right\}\right)^2\right\} \\ &= E\left\{\left(a_1 y_1 + a_2 y_2 - (a_1 C + a_2 C)\right)^2\right\} \\ &= E\left\{\left(a_1 e_1 + a_2 e_2\right)^2\right\}. \end{aligned}$$

Now substitute $a_2 = 1 - a_1$ in $var\{\widehat{C}\}$:

$$\begin{aligned} var\left\{\widehat{C}\right\} &= E\left\{\left(a_1\left(e_1 - e_2\right) + e_2\right)^2\right\} \\ &= a_1^2 E\left\{\left(e_1 - e_2\right)^2\right\} + 2a_1 E\left\{\left(e_1 - e_2\right) e_2\right\} + E\left\{e_2^2\right\}. \end{aligned}$$

Taking the derivative with respect to a_1 and setting it to zero gives

$$\begin{aligned} a_1 &= \frac{E\left\{\left(e_2 - e_1\right) e_2\right\}}{E\left\{\left(e_1 - e_2\right)^2\right\}} = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}, \\ a_2 &= \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}. \end{aligned}$$

The result is very intuitive. If $\sigma_2^2 \gg \sigma_1^2$, then the measurement y_2 is trusted less as $a_1 \approx 1$, $a_2 \approx 0$.

3. Define the augmented state $\mathbf{z}(n) = \left[\begin{array}{cc} \mathbf{x}(n) & \mathbf{v}(n) \end{array}\right]^{\mathrm{T}}$ which satisfies

$$\begin{aligned} \mathbf{z}(n) &= \left[\begin{array}{cc} \mathbf{A} & \mathbf{B}\mathbf{A}_v \\ \mathbf{0} & \mathbf{A}_v \end{array}\right] \mathbf{z}(n-1) + \left[\begin{array}{c} \mathbf{B}\mathbf{B}_v \\ \mathbf{B}_v \end{array}\right] \mathbf{e}(n), \\ \mathbf{y}(n) &= \left[\begin{array}{cc} \mathbf{C} & \mathbf{0} \end{array}\right] \mathbf{z}(n) + \mathbf{w}(n). \end{aligned}$$

4. The state-space representation is

$$\begin{aligned} x(n) &= x(n-1) = \alpha, \\ y(n) &= x(n) + w(n) \end{aligned}$$

with $E\{x(0)\} = 0$ and $E\{x(0)^2\} = \sigma_\alpha^2$.

In lectures we derived the Kalman filter:

$$\begin{aligned} \widehat{x}(n) &= \widehat{x}(n-1) + \frac{\sigma^2(n)}{\sigma^2(n) + \sigma_w^2}\left(y(n) - \widehat{x}(n-1)\right), \\ \sigma^2(n) &= \sigma^2(n-1)\left(1 - \frac{\sigma^2(n-1)}{\sigma^2(n-1) + \sigma_w^2}\right) \end{aligned}$$

with $\sigma^2(0) = \sigma_\alpha^2$. Note that $\sigma^2(n)$ is a positive sequence decreasing over time, i.e. $\sigma^2(n) < \sigma^2(n-1)$. Assume $\sigma^2(n)$ has a limit. Call the limit σ^2. Now solve

$$\sigma^2 = \sigma^2 \left(1 - \frac{\sigma^2}{\sigma^2 + \sigma_w^2}\right)$$

to get the answer, which is $\sigma^2 = 0$. Thus the Kalman filter converges towards the true value of the parameter.

5. Consider first the case when $p \geq q$:

$$\begin{aligned}
\mathbf{x}(n) &= \begin{bmatrix} \alpha(n) & \alpha(n-1) & \cdots & \alpha(n-p+1) \end{bmatrix}^{\mathrm{T}}, \\
\mathbf{F}(n) &= \begin{bmatrix} a_1 & a_2 & \cdots & a_p \\ 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{bmatrix}, \ \mathbf{G}(n) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \\
\mathbf{H}(n) &= \begin{bmatrix} b_0 & \cdots & b_{q-1} & \underbrace{0 \ \cdots \ 0}_{q-p} \end{bmatrix}.
\end{aligned}$$

The State and Observation Equation is:

$$\begin{aligned}
\mathbf{x}(n) &= \mathbf{F}(n)\,\mathbf{x}(n-1) + \mathbf{G}(n)\,v(n) \\
\mathbf{y}(n) &= \mathbf{H}(n)\,\mathbf{x}(n-1) + w(n)
\end{aligned}$$

Consider now the case where $p < q$ then

$$\begin{aligned}
\mathbf{x}(n) &= \begin{bmatrix} \alpha(n) & \alpha(n-1) & \cdots & \alpha(n-q+1) \end{bmatrix}^{\mathrm{T}}, \\
\mathbf{F}(n) &= \begin{bmatrix} a_1 & a_1 & \cdots & a_p & & & \\ 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & 1 & \end{bmatrix}, \ \mathbf{G}(n) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \\
\mathbf{H}(n) &= \begin{bmatrix} b_0 & \cdots & b_{q-1} \end{bmatrix}.
\end{aligned}$$