# 4F7 Examples Sheet 1 Corrections

1.

$$\mathbf{h}_{\text{opt}} = E^{-1}\left\{\mathbf{u}(n)\,\mathbf{u}^{\text{T}}(n)\right\} E\left\{\mathbf{u}(n)\,d(n)\right\} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}.$$

$$\begin{aligned} E\left\{e_{\min}^2\right\} &= E\left\{d^2(n)\right\} - \mathbf{h}_{\text{opt}}^{\text{T}} E\left\{\mathbf{u}(n)\,\mathbf{u}^{\text{T}}(n)\right\}\mathbf{h}_{\text{opt}} \\ &= 0.75. \end{aligned}$$

2. Autocorrelation function $E\{u(k)\,u(l)\}$:

$$\begin{aligned} \sigma_u^2 &= E\left\{u^2(n)\right\} = E\left\{(\alpha u(n-1) + v(n))^2\right\} \\ &= \alpha^2 E\left\{u^2(n-1)\right\} + E\left\{v^2(n)\right\} \\ &= \alpha^2\sigma_u^2 + \sigma_v^2 \\ &\Rightarrow \sigma_u^2 = \sigma_v^2/\left(1-\alpha^2\right). \end{aligned}$$

Then one has

$$E\{u(n)\,u(n-1)\} = \alpha E\left\{u^2(n-1)\right\} = \alpha\sigma_u^2.$$

And similarly one obtains

$$E\{u(k)\,u(l)\} = \alpha^{|k-l|}\sigma_u^2.$$

Cross-correlation function $E\{d(k)\,u(l)\}$:

$$\begin{aligned} E\{d(k)\,u(l)\} &= E\{(u(k) + w(k))\,u(l)\} \\ &= E\{u(k)\,u(l)\} + E\{w(k)\,u(l)\} \\ &= E\{u(k)\,u(l)\}. \end{aligned}$$

$\mathbf{h}_{\text{opt}}$ minimizing the mean square error:

$$\begin{aligned} \mathbf{h}_{\text{opt}} &= \left\{\sigma_u^2 \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}\right\}^{-1} \sigma_u^2 \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \\ &= \frac{1}{1-\alpha^2}\begin{pmatrix} 1 & -\alpha \\ -\alpha & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \alpha \end{pmatrix} \\ &= \frac{1}{1-\alpha^2}\begin{pmatrix} 1-\alpha^2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

(Given the model, the result is obvious.) Power of the residual error, i.e. $J(\mathbf{h}_{\text{opt}})$:

$$\begin{aligned}
E\left\{e_{\min}^2\right\} &= E\left\{d^2(n)\right\} - \mathbf{h}_{\text{opt}}^{\text{T}} E\left\{\mathbf{u}(n)\,\mathbf{u}^{\text{T}}(n)\right\}\mathbf{h}_{\text{opt}} \\
&= \sigma_w^2 + \sigma_u^2 - \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\text{T}} \sigma_u^2 \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \sigma_w^2 + \sigma_u^2 - \sigma_u^2 \\
&= \sigma_w^2.
\end{aligned}$$

Given the model, the result is obvious too.

3. Proving

$$\mathbf{A}^{-1} = \mu \sum_{k=0}^{\infty} (\mathbf{I} - \mu\mathbf{A})^k . \tag{1}$$

One has

$$\begin{aligned}
(\mathbf{I} - \mu\mathbf{A}) &= \left(\mathbf{I} - \mu\mathbf{Q}^{\text{T}}\mathbf{\Lambda}\mathbf{Q}\right) \\
&= \mathbf{Q}^{\text{T}}(\mathbf{I} - \mu\mathbf{\Lambda})\,\mathbf{Q}
\end{aligned}$$

and clearly

$$\begin{aligned}
&(\mathbf{I} - \mu\mathbf{A})^k = \mathbf{Q}^{\text{T}}(\mathbf{I} - \mu\mathbf{\Lambda})^k\,\mathbf{Q} \\
&\Rightarrow \textstyle\sum_{k=0}^{\infty}(\mathbf{I} - \mu\mathbf{A})^k = \mathbf{Q}^{\text{T}}\left(\sum_{k=0}^{\infty}(\mathbf{I} - \mu\mathbf{\Lambda})^k\right)\mathbf{Q}.
\end{aligned}$$

The matrix $(\mathbf{I} - \mu\mathbf{\Lambda})^k$ is a diagonal matrix of $i^{\text{th}}$ diagonal element given by $(1 - \mu\lambda_i)^k$. As the matrix is definite positive, one has $\lambda_i > 0$ and if $\mu$ is small enough then $|1 - \mu\lambda_i| < 1$. It follows that

$$\sum_{k=0}^{\infty}(1 - \mu\lambda_i)^k = \frac{1}{1 - (1 - \mu\lambda_i)} = \frac{1}{\mu\lambda_i}$$

thus

$$\begin{aligned}
\mu\mathbf{Q}^{\text{T}}\left(\sum_{k=0}^{\infty}(\mathbf{I} - \mu\mathbf{\Lambda})^k\right)\mathbf{Q} &= \mu \times \mu^{-1}\mathbf{Q}^{\text{T}}\mathbf{\Lambda}^{-1}\mathbf{Q} \\
&= \mathbf{A}^{-1}.
\end{aligned}$$

Establishing the convergence of the SD algorithm initialized at $\mathbf{h}(0) = \mathbf{p}$:

$$\begin{aligned}
\mathbf{h}_{\text{opt}} &= \mathbf{R}^{-1}\mathbf{p} \\
&= \mu \sum_{k=0}^{\infty}(\mathbf{I} - \mu\mathbf{R})^k\,\mathbf{p}.
\end{aligned}$$

It can be checked easily that the SD algorithm initialized with $\mathbf{h}(0) = \mathbf{p}$ is a simple algorithm computing at time $n$

$$\mathbf{h}(n) = \mu \sum_{k=0}^{n}(\mathbf{I} - \mu\mathbf{R})^k\,\mathbf{p}.$$

Indeed one has

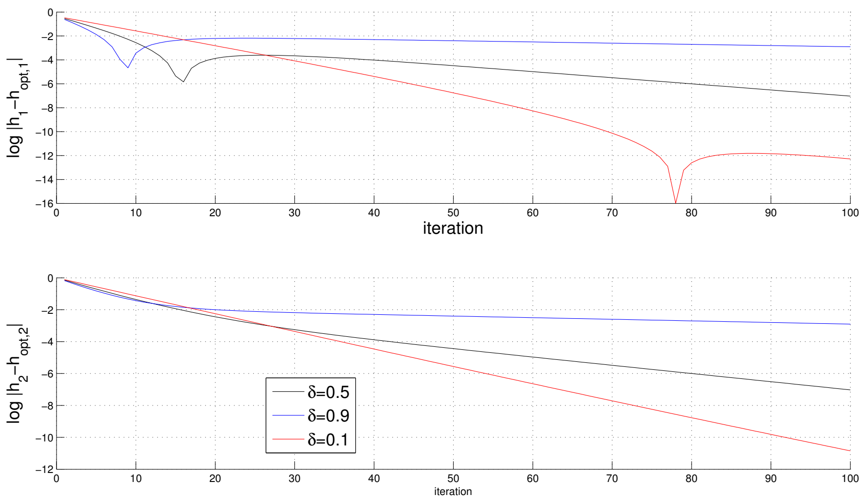
$$\begin{aligned}\mathbf{h}(n) &= \mathbf{h}(n-1) + \mu\left[\mathbf{p} - \mathbf{R}\mathbf{h}(n-1)\right] \\ &= (\mathbf{I} - \mu\mathbf{R})\,\mathbf{h}(n-1) + \mu\mathbf{p} \\ &= \mu\sum_{k=0}^{n-1}(\mathbf{I} - \mu\mathbf{R})^{k+1}\,\mathbf{p} + \mu\mathbf{p} \\ &= \mu\sum_{k=0}^{n}(\mathbf{I} - \mu\mathbf{R})^{k}\,\mathbf{p}.\end{aligned}$$

4.
$$\det(\mathbf{R} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & \delta \\ \delta & 1-\lambda \end{vmatrix} = (\lambda - 1)^2 - \delta^2.$$

It follows that the eigenvalues are given by $\lambda_{\min} = 1 - \delta$ and $\lambda_{\max} = 1 + \delta$. As $\delta \to 1$, one has

$$\frac{\lambda_{\max}}{\lambda_{\min}} \to \infty$$

and the rate of convergence of SD decreases.
Here are the plots for $\mathbf{h}(0) = \begin{bmatrix} 0.3 & 0 \end{bmatrix}^{\mathrm{T}}$, $\mu = 0.1$ and $\delta \in \{0.9, 0.5, 0.1\}$.




5. In both cases, we need to compute $E\left\{u^2(n)\right\}$. For the FIR filter, one has

$$\begin{aligned}E\left\{u^2(n)\right\} &= E\left\{\left(\sum_{i=0}^{L-1}\alpha_i v(n-i)\right)^2\right\} \\ &= \sum_{i=0}^{L-1}\alpha_i^2 E\left\{v^2(n-i)\right\} \\ &= \sigma_v^2\left(\sum_{i=0}^{L-1}\alpha_i^2\right).\end{aligned}$$

For the IIR filter, one has

$$
\begin{aligned}
&E\left(u^2(n)\right) - a_1 E(u(n)u(n-1)) - a_2 E(u(n)u(n-2)) = E(u(n)v(n)) \\
&\Leftrightarrow r_u(0) - a_1 r_u(1) - a_2 r_u(2) = \sigma_w^2; \\
&E(u(n-1)u(n)) - a_1 E(u(n-1)u(n-1)) - a_2 E(u(n-1)u(n-2)) = E(u(n-1)v(n)) \\
&\Leftrightarrow r_u(1) - a_1 r_u(0) - a_2 r_u(1) = 0, \\
&E(u(n-2)u(n)) - a_1 E(u(n-2)u(n-1)) - a_2 E(u(n-2)u(n-2)) = E(u(n-2)v(n)) \\
&\Leftrightarrow r_u(2) - a_1 r_u(1) - a_2 r_u(0) = 0.
\end{aligned}
$$

It follows that

$$
\begin{bmatrix} 1 & -a_1 & -a_2 \\ -a_1 & 1-a_2 & 0 \\ -a_2 & -a_1 & 1 \end{bmatrix}
\begin{bmatrix} r_u(0) \\ r_u(1) \\ r_u(2) \end{bmatrix}
=
\begin{bmatrix} \sigma_v^2 \\ 0 \\ 0 \end{bmatrix}.
$$

Solving for $r_u(0)$, $r_u(1)$ and $r_u(2)$ one obtains

$$
r_u(0) = E\left\{u^2(n)\right\} = \left(\frac{1-a_2}{1+a_2}\right)\frac{\sigma_w^2}{(1-a_2)^2 - a_1^2}.
$$