4F7 Examples Sheet 1

1. Consider the following Wiener filtering problem:

$$E\left\{\mathbf{u}(n)\,\mathbf{u}^{\mathrm{T}}(n)\right\} = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix},\ E\left\{\mathbf{u}(n)\,d(n)\right\} = \begin{pmatrix} 0.5 \\ 0.25 \end{pmatrix}$$

and $E\left\{d^2(n)\right\} = 1$.

- Calculate the Wiener filter.
- What is the minimum mean square error produced by this filter if $E\left\{d^2(n)\right\} = 1$?

2. Consider the following signal model

$$u(n) = \alpha u(n-1) + v(n)$$

where $|\alpha| < 1$, $\{v(n)\}$ is a zero-mean i.i.d. (independent and identically distributed) noise sequence with $E\left\{v^2(n)\right\} = \sigma_v^2$. Let

$$d(n) = u(n) + w(n)$$

where$\{w(n)\}$ is a zero-mean i.i.d. noise sequence with $E\left\{w^2(n)\right\} = \sigma_w^2$. The noise $\{w(n)\}$ is statistically independent of $\{u(n)\}$.

- Compute the autocorrelation function $E\{u(k)\,u(l)\}$ and the cross-correlation function $E\{d(k)\,u(l)\}$.
- Let

$$\mathbf{h}_{\mathrm{opt}} = \arg\min_{\mathbf{h}} J(\mathbf{h})$$

where

$$J(\mathbf{h}) = E\left\{\left(d(n) - \mathbf{h}^{\mathrm{T}}\mathbf{u}(n)\right)^2\right\}$$

with $\mathbf{h} = \begin{bmatrix} h_0 & h_1 \end{bmatrix}^{\mathrm{T}}$ and $\mathbf{u}(n) = \begin{bmatrix} u(n) & u(n-1) \end{bmatrix}^{\mathrm{T}}$. What is $\mathbf{h}_{\mathrm{opt}}$?

- What is the power of the residual error, i.e. $J(\mathbf{h}_{\mathrm{opt}})$?

3. The Steepest Descent (SD) algorithm is a gradient algorithm minimizing

$$J(\mathbf{h}) = E\left\{\left(d(n) - \mathbf{h}^{\mathrm{T}}\mathbf{u}(n)\right)^2\right\}.$$

- Show that for any symmetric positive definite matrix $\mathbf{A}$, provided μ is small enough,

$$\mathbf{A}^{-1} = \mu \sum_{k=0}^{\infty} (\mathbf{I} - \mu \mathbf{A})^k . \qquad (1)$$

(Hint: Use the eigendecomposition of $\mathbf{A}$; i.e. $\mathbf{A} = \mathbf{Q}^{\mathrm{T}} \mathbf{\Lambda} \mathbf{Q}$ where $\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{I}$ and $\mathbf{\Lambda}$ is diagonal).

- Assume the covariance matrix $\mathbf{R} = E\left\{\mathbf{u}(n)\,\mathbf{u}^{\mathrm{T}}(n)\right\}$ is positive definite and let $\mathbf{p} = E\{d(n)\,\mathbf{u}(n)\}$. Using (1), show that the SD algorithm will converge when initialized at $\mathbf{h}(0) = \mathbf{p}$.

4. (Matlab) Convergence speed of the steepest descent algorithm is dictated by the eigenvalue spread of the correlation matrix $\mathbf{R}$. Consider the case where

$$\mathbf{R} = E\left\{\mathbf{u}(n)\,\mathbf{u}^{\mathrm{T}}(n)\right\} = \begin{pmatrix} 1 & \delta \\ \delta & 1 \end{pmatrix}, \ \mathbf{p} = E\{\mathbf{u}(n)\,d(n)\} = \begin{pmatrix} 1+\delta \\ 1+\delta \end{pmatrix}.$$

Compute the eigenvalues of $\mathbf{R}$. Initialize the algorithm with $\mathbf{h}(0)$ and iterate until the algorithm converges. Try different values for δ, $0 < \delta < 1$, and μ. What happens?

5. Consider the following signals

L-tap FIR	$u(n) = \sum_{i=0}^{L-1} \alpha_i v(n-i)$
2-tap IIR	$u(n) = a_1 u(n-1) + a_2 u(n-2) + v(n)$

where $\{v(n)\}$ is a zero-mean i.i.d. noise sequence with $E\left\{v^2(n)\right\} = \sigma_v^2$.

If these signals are the input of a LMS filter of length M, what is the stability limit on the stepsize μ given by $\left(ME\left\{u^2(n)\right\}\right)^{-1}$ for these two signals?