4F5 Advanced Communications and Coding
Michealmas 2013

Examples Paper 3

Question 1

Differential entropy: Evaluate the differential entropy $h(.)$ for the following distributions:

(a) The exponential density, $f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$

(b) The Laplacian density, $f(x) = \frac{1}{2}\lambda e^{-\lambda|x|}, \quad x \in \mathbb{R}$

(c) The sum of X_1 and X_2, where X_1 and X_2 are independent Gaussian random variables distributed as $\mathcal{N}(\mu_1, \sigma_1^2)$ and $\mathcal{N}(\mu_2, \sigma_2^2)$, respectively.

Question 2

The Gaussian maximises differential entropy: Let X be any continuous random variable with mean 0 and variance σ^2. Then show that

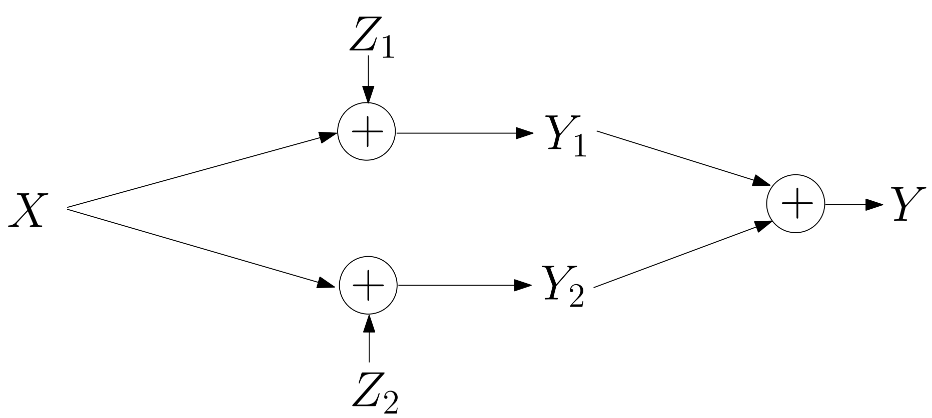
$$h(X) \leq \frac{1}{2}\log 2\pi e\sigma^2$$

with equality if and only if X is distributed as $\mathcal{N}(0, \sigma^2)$.

Hint: Use $0 \leq D(f||\phi)$, where f is the density of X, and ϕ is the density of a $\mathcal{N}(0, \sigma^2)$ random variable.

Question 3

Multipath Gaussian channel with no delays: Consider a discrete-time Gaussian noise channel with input power constraint P, where the each input symbol takes two different paths and the received noisy signals are added together at the receiver antenna, as shown below.



(a) Verify that the channel above reduces to the following channel: $Y = 2X + Z_1 + Z_2$

(b) Find the capacity of this channel if Z_1 and Z_2 are jointly Gaussian with zero mean and covariance matrix given by

$$K_{Z_1Z_2} = \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix}$$

(c) What is the capacity when $\rho = 0$, $\rho = 1$, $\rho = -1$?

Question 4

Detection with non-uniform symbol probabilities: Consider BPSK modulation with symbols $\{+A, -A\}$ over the discrete-time AWGN channel

$$Y = X + N$$

where N is Gaussian noise $\sim \mathcal{N}(0, N_0/2)$. Suppose that $P(X = A) = p$ and $P(X = -A) = 1 - p$.

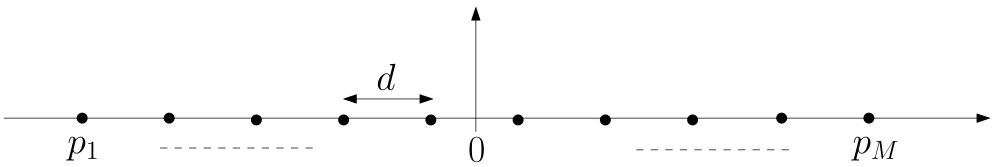
(a) Derive the detection rule that that minimises the probability of detection error. Sketch the decision regions when $p = 2/3$ and $A/N_0 = 4$.

(b) Obtain the average probability of detection error, first in terms of p, A, N_0, then express in terms of p and E_b/N_0.

Question 5

M-ary Pulse Amplitude Modulation (PAM): Consider the M-ary PAM constellation shown in the figure below. It consists of M symbols $\{p_1, \ldots, p_M\}$ on the real line, symmetric around 0 and with equal spacing d between symbols. That is,

$$p_i = (2i - 1 - M)\frac{d}{2}, \quad i = 1, \ldots, M$$



Suppose that we use this constellation to signal over the discrete-time AWGN channel

$$Y = X + N$$

where the Gaussian noise N is distributed $\sim \mathcal{N}(0, N_0/2)$. Assuming all the constellation symbols are equally likely:

(a) Sketch the decision regions that minimise the probability of detection error.

(b) Obtain the probability of error when p_1 or p_M is sent.

(c) Obtain the probability of error when p_i is sent, for $2 \leq i \leq M - 1$. Combine this with part (b) to obtain an expression for the overall probability of error P_e.

(d) Show that the average symbol energy E_s is $\frac{(M^2-1)d^2}{12}$. (Induction may be useful)

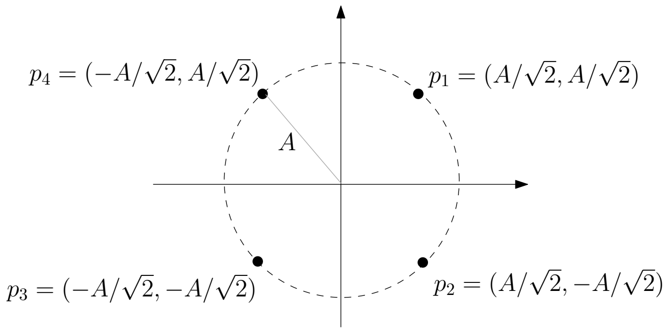
(e) Express the probability of error P_e in terms of $\frac{E_b}{N_0}$. For fixed E_b/N_0, how does P_e vary as M increases? Is this what you'd expect?

Question 6

Quadrature Phase Shift Keying: Consider QPSK modulation over an AWGN channel

$$Y = X + N$$

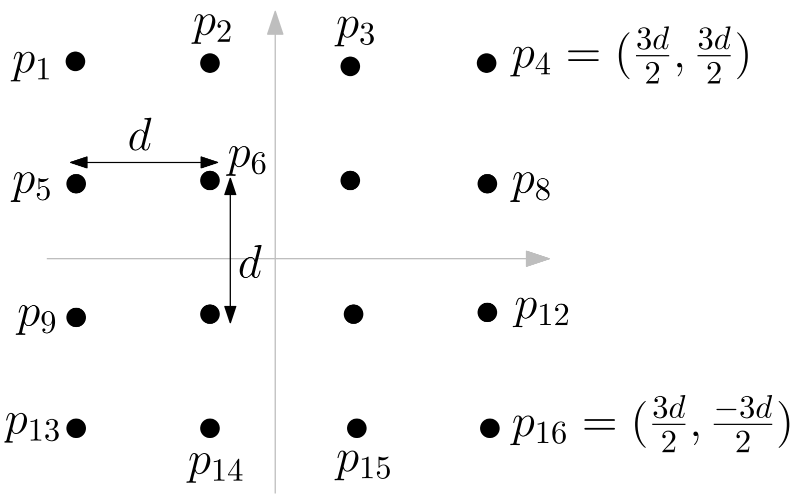
where the noise N is a complex random variable distributed as $\mathcal{CN}(0, N_0)$, i.e., the real and imaginary parts of N are i.i.d Gaussian $\sim \mathcal{N}(0, N_0/2)$. X is a symbol drawn uniformly from the QPSK constellation below.



Sketch the optimal decision regions, and show the probability of detection error $P_e \leq 2\mathcal{Q}\left(\sqrt{\frac{2E_b}{N_0}}\right)$. (Handout 7, slide 27 may be helpful.)

Question 7

Quadrature Amplitude Modulation: Consider the 16-QAM constellation shown in the figure below, with adjacent symbols in the vertical and horizontal directions spaced d apart.



This constellation is used for signalling (with uniform distribution on the symbols) over the AWGN channel

$$Y = X + N.$$

The noise N is a complex random variable distributed as $\mathcal{CN}(0, N_0)$, i.e., the real and imaginary parts of N are i.i.d Gaussian $\sim \mathcal{N}(0, N_0/2)$.

(a) Derive an upper bound for the probability of error when $X = p_1$ (or $X = p_4/p_{13}/p_{16}$, one of the corner points of the constellation).

(b) Derive an upper bound for the probability of error when $X = p_2$.

(c) Derive an upper bound for the probability of error when $X = p_6$.

(d) Using the union bound show that the average probability of error satisfies

$$P_e \leq 3\mathcal{Q}\left(\frac{d}{\sqrt{2N_0}}\right) = 3\mathcal{Q}\left(\sqrt{\frac{4E_b}{5N_0}}\right)$$

where E_b is the average energy per bit of the constellation. (For the last equality, you'll first need to show that $E_s = 2.5d^2$)

Question 8

BPSK over a Rayleigh Flat Fading channel: In Handout 9, we showed that the probability of error for BPSK over a fading channel with coherent detection is given by

$$P_e = \mathbb{E}\left[\mathcal{Q}\left(\sqrt{2|h|^2\,\mathrm{snr}}\right)\right] \qquad \text{where snr} = \frac{E_b}{N_0}. \tag{1}$$

Recall that $|h|^2$, the squared-magnitude of the fading coefficient h has an exponential density f:

$$f(x) = \exp\left(-x\right), \quad x \geq 0.$$

Show that the average error probability in (1) is equal to $\frac{1}{2}\left(1 - \sqrt{\frac{\mathrm{snr}}{1+\mathrm{snr}}}\right)$.

(Hint: Write the expression in (1) as a double integral and interchange the order of integration.)

Question 9

Diversity via Repetition coding: Consider the fading channel

$$Y = hX + N$$

In Handout 9, we saw how repetition coding can be used to improve the error performance of BPSK on the fading channel. Here we explore repetition coding with QPSK symbols. Consider L uses of the channel above to transmit a symbol x drawn uniformly from the QPSK constellation shown in Question 6. The output vector is

$$\mathbf{Y} = \mathbf{h}x + \mathbf{N}$$

where $\mathbf{h} = (h[1], \ldots, h[L])^T$ is a vector of complex Gaussian rvs that are i.i.d $\sim \mathcal{CN}(0, 1)$. (We assume that there is interleaving so that the L uses of the channel are over different coherence periods.) $\mathbf{N} = (N[1], N[2], \ldots, N[m])^T$ is a vector of complex Gaussian rvs that are i.i.d $\sim \mathcal{CN}(0, N_0)$.

We now perform coherent detection.

(a) Project $\mathbf{Y}$ along the direction of $\mathbf{h}$, and observe that the problem reduces to an instance of QPSK detection in AWGN. Write down or sketch the decision regions.

(b) Show that the probability of error conditioned on $\mathbf{h}$ is upper bounded by $2\mathcal{Q}\left(\sqrt{\frac{2\|\mathbf{h}\|^2 E_b}{N_0}}\right)$.

(c) The $\mathcal{Q}$ function can be upper bounded as $\mathcal{Q}(x) < \frac{1}{2}e^{-x^2/2}$ for $x > 0$. Use this to show that

$$P_{e|\mathbf{h}} \leq \prod_{m=1}^{L} e^{-\frac{E_b}{N_0}|h[m]|^2}$$

(d) Show that the average probability of error is

$$P_e < \left(1 + \frac{E_b}{N_0}\right)^{-L}$$

Hint: Use the fact that the rvs $|h[m]|^2$ for $m = 1, \ldots, L$ are i.i.d with exponential density $f(x) = e^{-x},\ x \geq 0$

(e) (Extra) Repeat the steps above assuming x came from a 4-PAM constellation $\left\{-\frac{3d}{2}, -\frac{d}{2}, \frac{d}{2}, \frac{3d}{2}\right\}$. Show that the average probability of error is upper bounded by

$$P_e < \frac{3}{4}\left(1 + \frac{2}{5}\frac{E_b}{N_0}\right)^{-L}.$$

(Your calculations for Question 5 will be useful)

Thus QPSK has better error performance than 4-PAM as E_b/N_0 gets large though both transmit 2 bits per symbol. This is because QPSK uses two dimensions, while PAM packs all four symbols along the same dimension.

Answers

Q1. (a) $\log\left(\frac{e}{\lambda}\right)$ bits; (b) $\log\left(\frac{2e}{\lambda}\right)$ bits; (c) $\frac{1}{2}\log\left(2\pi e(\sigma_1^2 + \sigma_2^2)\right)$ bits

Q3. (b) $\mathcal{C} = \frac{1}{2}\log\left(1 + \frac{2P}{\sigma^2(1+\rho)}\right)$; (c) When $\rho = 0$, $\mathcal{C} = \frac{1}{2}\log\left(1 + \frac{2P}{\sigma^2}\right)$. When $\rho = 1$, $\mathcal{C} = \frac{1}{2}\log\left(1 + \frac{P}{\sigma^2}\right)$. When $\rho = -1$, $\mathcal{C} = \infty$.

Q4. (a) Decode $\hat{X} = A$ when $Y \geq T$ and $\hat{X} = -A$ when $Y < T$, where the threshold $T = \frac{4N_0}{A}\ln\left(\frac{1-p}{p}\right)$. Note that $T = 0$, when $p = \frac{1}{2}$.

(c) $P_e = p\,\mathcal{Q}\left(\frac{A-T}{\sqrt{N_0/2}}\right) + (1-p)\,\mathcal{Q}\left(\frac{A+T}{\sqrt{N_0/2}}\right)$; $E_b = A^2$

Q5. (b) $\mathcal{Q}\left(\frac{d}{\sqrt{2N_0}}\right)$; (c) $2\mathcal{Q}\left(\frac{d}{\sqrt{2N_0}}\right)$, overall $P_e = \frac{2(M-1)}{M}\mathcal{Q}\left(\frac{d}{\sqrt{2N_0}}\right)$

Q7. (a) $2\mathcal{Q}\left(\frac{d}{\sqrt{2N_0}}\right)$; (b) $3\mathcal{Q}\left(\frac{d}{\sqrt{2N_0}}\right)$; (c) $4\mathcal{Q}\left(\frac{d}{\sqrt{2N_0}}\right)$