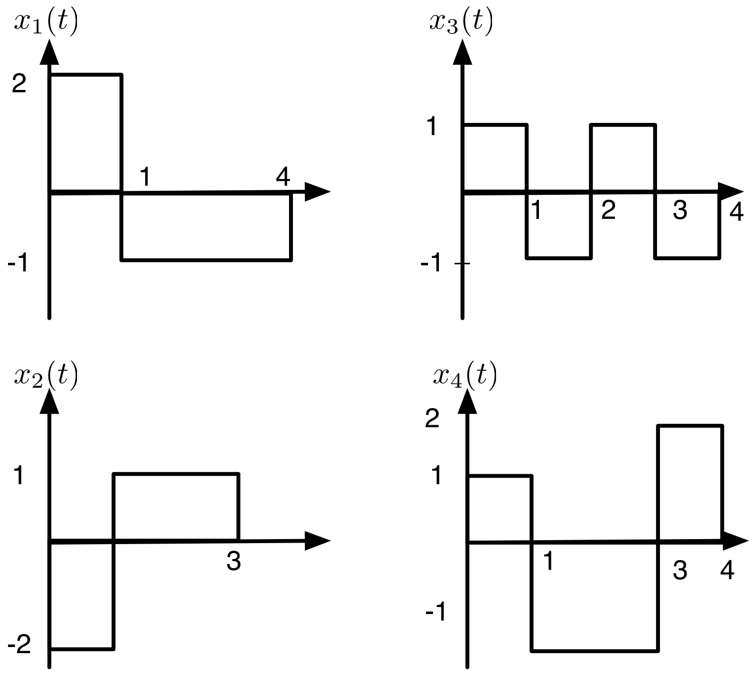
4F5 Advanced Communications and Coding
Michealmas 2015
Examples Paper 3

Question 1

Signal Space: Consider the four waveforms $x_i(t)(\cdot), \ldots, x_4(t)$ shown below.



(a) Determine the dimensionality of the waveforms and a set of orthonormal basis functions.

(b) Use the basis functions to represent the four waveforms by vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$.

(c) The distance between any two waveforms $x_i(t), x_j(t)$ can be defined as

$$d_{ij} = \left(\int (x_i(t) - x_j(t))^2 dt \right)^{\frac{1}{2}}.$$

Show that $d_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|$. (Note that $\|\mathbf{x}\|$ denotes the Euclidean norm of the vector $\mathbf{x}$.)

(d) Use part (c) to determine the minimum distance between any pair of waveforms shown above.

Question 2

Detection with non-uniform symbol probabilities: Consider BPSK modulation with symbols $\{+A, -A\}$ over the discrete-time AWGN channel

$$Y = X + N$$

where N is Gaussian noise $\sim \mathcal{N}(0, N_0/2)$. Suppose that $P(X = A) = p$ and $P(X = -A) = 1 - p$.

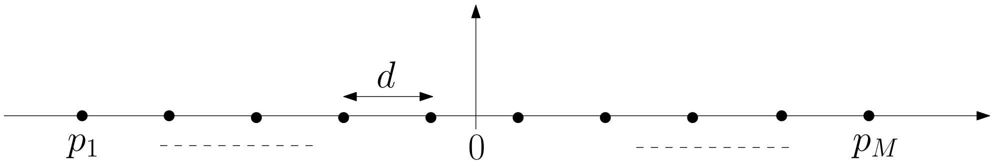
(a) Derive the detection rule that that minimises the probability of detection error. Sketch the decision regions when $p = 2/3$ and $A/N_0 = 4$.

(b) Obtain the average probability of detection error, first in terms of p, A, N_0, then express in terms of p and E_b/N_0.

Question 3

M-ary Pulse Amplitude Modulation (PAM): Consider the M-ary PAM constellation shown in the figure below. For $M \geq 2$, the constellation consists of M symbols $\{p_1, \dots, p_M\}$ on the real line, symmetric around 0 and with equal spacing d between symbols. That is,

$$p_i = (2i - 1 - M)\frac{d}{2}, \quad i = 1, \dots, M$$



Suppose that we use this constellation to signal over the discrete-time AWGN channel

$$Y = X + N$$

where the Gaussian noise N is distributed $\sim \mathcal{N}(0, N_0/2)$. Assuming all the constellation symbols are equally likely:

(a) Sketch the decision regions that minimise the probability of detection error.

(b) Obtain the probability of error when p_1 or p_M is sent.

(c) Obtain the probability of error when p_i is sent, for $2 \leq i \leq M - 1$. Combine this with part (b) to obtain an expression for the overall probability of error P_e.

(d) Show that the average symbol energy E_s is $\frac{(M^2-1)d^2}{12}$. (Induction may be useful)

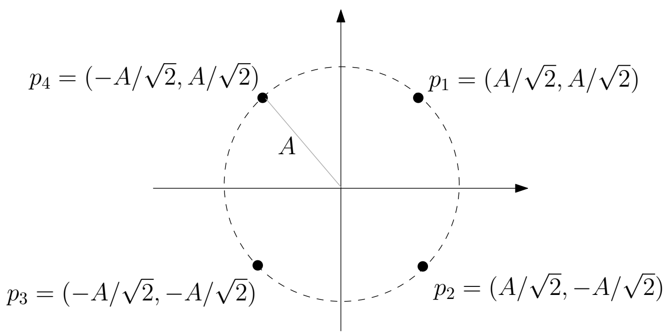
(e) Express the probability of error P_e in terms of $\frac{E_b}{N_0}$. For fixed E_b/N_0, how does P_e vary as M increases? Is this what you'd expect?

Question 4

Quadrature Phase Shift Keying: Consider QPSK modulation over an AWGN channel

$$Y = X + N$$

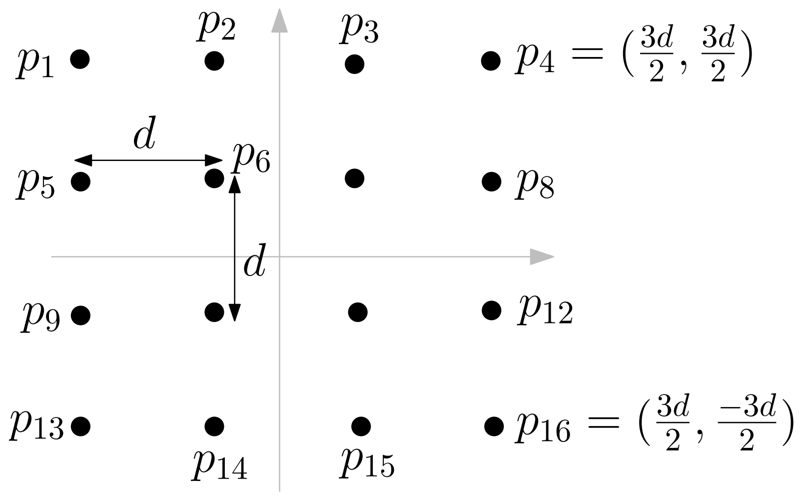
where the noise N is a complex random variable distributed as $\mathcal{CN}(0, N_0)$, i.e., the real and imaginary parts of N are i.i.d. Gaussian $\sim \mathcal{N}(0, N_0/2)$. X is a symbol drawn uniformly from the QPSK constellation below.



Sketch the optimal decision regions, and show the probability of detection error $P_e \leq 2\mathcal{Q}\left(\sqrt{\frac{2E_b}{N_0}}\right)$. (The main steps involved in computing P_e are outlined in Handout 7.)

Question 5

Quadrature Amplitude Modulation: Consider the 16-QAM constellation shown in the figure below, with adjacent symbols in the vertical and horizontal directions spaced d apart.



This constellation is used for signalling (with uniform distribution on the symbols) over the AWGN channel

$$Y = X + N.$$

The noise N is a complex random variable distributed as $\mathcal{CN}(0, N_0)$, i.e., the real and imaginary parts of N are i.i.d. Gaussian $\sim \mathcal{N}(0, N_0/2)$.

(a) Derive an upper bound for the probability of error when $X = p_1$ (or $X = p_4/p_{13}/p_{16}$, one of the corner points of the constellation).

(b) Derive an upper bound for the probability of error when $X = p_2$.

(c) Derive an upper bound for the probability of error when $X = p_6$.

(d) Using the union bound show that the average probability of error satisfies

$$P_e \leq 3\mathcal{Q}\left(\frac{d}{\sqrt{2N_0}}\right) = 3\mathcal{Q}\left(\sqrt{\frac{4E_b}{5N_0}}\right)$$

where E_b is the average energy per bit of the constellation. (For the last equality, you'll first need to show that $E_s = 2.5d^2$)

Question 6

M-ary FSK: After demodulation, an M-ary FSK receiver has the length-M vector $\mathbf{Y}$, given by

$$\mathbf{Y} = \mathbf{X} + \mathbf{N},$$

where $N_i, \ldots, N_s$ are i.i.d. Gaussian $\sim \mathcal{N}(0, N_0/2)$. If message i was transmitted, $\mathbf{X}$ has $\sqrt{E_s}$ in the ith entry and zeros elsewhere. Note that $E_s = E_b \log_2 M$ is the transmitted energy per symbol.

(a) Derive the optimal detection rule for the M-ary FSK receiver.

(b) Show that the probability of detection error can be bounded as $P_e \leq e^{-(\log_2 M)(\frac{E_b}{N_0} - 2\ln 2)}$. (The main steps are outlined in Handout 7. You also need to use the bound $\mathcal{Q}(x) < \frac{1}{2}e^{-x^2/2}$ for $x > 0$.)

(c) Compare the bandwidth efficiency (rate/bandwidth) of M-ary FSK with M-ary QAM assuming that the bandwidth of the QAM signal is $2W$ where $W = \frac{1}{T}$. Can you give an intuitive explanation for why QAM is more bandwidth efficient than FSK as M grows large?

(d) How do the probabilities of detection error for the two modulation schemes (M-QAM and M-FSK) compare as M grows large? (Hint: using a union bound, show that the probability of error for any symbol of square M-QAM constellation (like in Q.5) can be bounded by $4\mathcal{Q}(\frac{d}{\sqrt{2N_0}})$; then use the fact that $d^2 = \kappa E_s = \kappa E_b \log_2 M$ for some constant κ.)

Question 7

BPSK over a Rayleigh Flat Fading channel: In Handout 10, we showed that the probability of error for BPSK over a fading channel with coherent detection is given by

$$P_e = \mathbb{E}\left[\mathcal{Q}\left(\sqrt{2|h|^2\,\mathrm{snr}}\right)\right] \qquad \text{where snr} = \frac{E_b}{N_0}. \tag{1}$$

Recall that $|h|^2$, the squared-magnitude of the fading coefficient h has an exponential density f:

$$f(x) = \exp(-x), \quad x \geq 0.$$

Show that the average error probability in (1) is equal to $\frac{1}{2}\left(1 - \sqrt{\frac{\mathrm{snr}}{1+\mathrm{snr}}}\right)$.

(Hint: Write the expression in (1) as a double integral and interchange the order of integration.)

Question 8

Diversity via Repetition coding: Consider the fading channel

$$Y = hX + N$$

In Handout 10, we saw how repetition coding can be used to improve the error performance of BPSK on the fading channel. Here we explore repetition coding with QPSK symbols. Consider L uses of the channel above to transmit a symbol x drawn uniformly from the QPSK constellation shown in Question 6. The output vector is

$$\mathbf{Y} = \mathbf{h}x + \mathbf{N}$$

where $\mathbf{h} = (h[1], \ldots, h[L])^T$ is a vector of complex Gaussian rvs that are i.i.d. $\sim \mathcal{CN}(0, 1)$. (We assume that there is interleaving so that the L uses of the channel are over different coherence periods.) $\mathbf{N} = (N[1], N[2], \ldots, N[m])^T$ is a vector of complex Gaussian rvs that are i.i.d. $\sim \mathcal{CN}(0, N_0)$.

We now perform coherent detection.

(a) Project $\mathbf{Y}$ along the direction of $\mathbf{h}$, and observe that the problem reduces to an instance of QPSK detection in AWGN. Write down or sketch the decision regions.

(b) Show that the probability of error conditioned on $\mathbf{h}$ is upper bounded by $2\mathcal{Q}\left(\sqrt{\frac{2\|\mathbf{h}\|^2 E_b}{N_0}}\right)$.

(c) The $\mathcal{Q}$ function can be upper bounded as $\mathcal{Q}(x) < \frac{1}{2}e^{-x^2/2}$ for $x > 0$. Use this to show that

$$P_{e|\mathbf{h}} \leq \prod_{m=1}^{L} e^{-\frac{E_b}{N_0}|h[m]|^2}$$

(d) Show that the average probability of error is

$$P_e < \left(1 + \frac{E_b}{N_0}\right)^{-L}$$

Hint: Use the fact that the rvs $|h[m]|^2$ for $m = 1, \ldots, L$ are i.i.d. with exponential density $f(x) = e^{-x},\ x \geq 0$

(e) Repeat the steps above assuming x came from a 4-PAM constellation $\left\{-\frac{3d}{2}, -\frac{d}{2}, \frac{d}{2}, \frac{3d}{2}\right\}$. Show that the average probability of error is upper bounded by

$$P_e < \frac{3}{4}\left(1 + \frac{2}{5}\frac{E_b}{N_0}\right)^{-L}.$$

(Your calculations for Question 5 will be useful)

Thus QPSK has better error performance than 4-PAM as E_b/N_0 gets large though both transmit 2 bits per symbol. This is because QPSK uses two dimensions, while PAM packs all four symbols along the same dimension.

Question 9

Diversity via multiple Transmit Antennas:

Answers

Q1. (d) The minimum distance between waveforms is $\sqrt{5}$.

Q2. (a) Decode $\hat{X} = A$ when $Y \geq T$ and $\hat{X} = -A$ when $Y < T$, where the threshold $T = \frac{N_0}{4A} \ln\left(\frac{1-p}{p}\right)$. Note that $T = 0$, when $p = \frac{1}{2}$.

(c) $P_e = p\,\mathcal{Q}\left(\frac{A-T}{\sqrt{N_0/2}}\right) + (1-p)\,\mathcal{Q}\left(\frac{A+T}{\sqrt{N_0/2}}\right)$; $E_b = A^2$

Q3. (b) $\mathcal{Q}\left(\frac{d}{\sqrt{2N_0}}\right)$; (c) $2\mathcal{Q}\left(\frac{d}{\sqrt{2N_0}}\right)$, overall $P_e = \frac{2(M-1)}{M}\mathcal{Q}\left(\frac{d}{\sqrt{2N_0}}\right)$

Q5. (a) $2\mathcal{Q}\left(\frac{d}{\sqrt{2N_0}}\right)$; (b) $3\mathcal{Q}\left(\frac{d}{\sqrt{2N_0}}\right)$; (c) $4\mathcal{Q}\left(\frac{d}{\sqrt{2N_0}}\right)$