3F3 Example Paper 1
DFT, FFT, Digital Filters

1. (Revision). Show the following properties of the DFT of an N-point data sequence $\{x_n\}$:

   (a) Periodic spectrum - i.e. $X_{N+p} = X_p$

   (b) Periodic data - i.e. $x_{n+N} = x_n$ [Hint: use inverse DFT expression to calculate both x_{n+N} and x_n.]

   (c) Conjugate symmetry for real-valued $\{x_n\}$, i.e. $X_k = X^*_{N-k}$

2. Let X_k be the $N-$point DFT of the sequence x_n, $0 \leq n \leq N-1$. What is the $N-$point DFT of the sequence $s_n = X_n$, $0 \leq n \leq N-1$?

3. Suppose that an FFT hardware unit is available for computing the DFT. Show how the same unit may be used without modification to compute an Inverse DFT (by performing suitable manipulations of the input and output vectors).

4. Given that the N-point DFT X_k of a real-valued sequence x_n, $n = 0, 1, \ldots, N$ has the conjugate symmetry property

$$X_{N-k} = X^*_k$$

   deduce how a single N-point FFT may be used to compute the N-point DFTs of two length N real-valued data sequences y_n and z_n simultaneously by first forming the complex sequence $x_n = y_n + jz_n$.

5. Starting with the 3-point DFT $k = 0,\ 1,\ 2,$

$$X_k = \sum_{n=0}^{2} x_n e^{-j\frac{2\pi}{3}kn}$$

   - derive a nine-point (3 x 3) FFT algorithm and draw its flow diagram. Hint: use the same technique as the radix-2 derivation, but split the 9-point DFT into 3 interleaved 3-point DFTs, etc.
   - calculate how many complex multiplications (excluding multiplications by unity) and additions are required, and compare this with the number required for a direct evaluation of the 9-point DFT.

6. Compute the frequency responses of the FIR filters with the following impulse responses, exploiting symmetry to express each frequency response as the product of a pure delay term and a frequency-dependent gain. State what type of filter (eg. highpass, lowpass, etc.) each is

   (a) 1, 2, 1
   (b) -1, 2, -1
   (c) -1, 0, 2, 0, -1
   (d) 1, 2, 2, 1

7. It is required to design an FIR bandpass filter of order 200. $H_d(\omega)$ represents the ideal characteristic of the noncausal bandpass filter defined by

$$H_d(\omega) = \begin{cases} 1 & \text{if } 0.4\pi < |\omega| < 0.5\pi \\ 0 & \text{otherwise} \end{cases}$$

   (a) Determine the impulse response $h_d(n)$ corresponding to $H_d(\omega)$.
   (b) Explain how you would use the Hamming window

$$w_n = 0.54 + 0.46\cos\left(\frac{2\pi}{M-1}\right), \quad -\frac{M-1}{2} \leq n \leq \frac{M-1}{2}$$

   to design a causal FIR bandpass filter which approximates the ideal bandpass filter, having an impulse response h_n for $0 \leq n \leq 200$
   (c) Discuss the advantages/disadvantages of using a Hamming window compared to a rectangular window, including ripple amplitude, transition bandwidth and stop-band attenuation.

8. For an audio system with sampling rate 44.1 kHz, a bandpath filter is required with 3dB corner frequencies at 6.5084 kHz and 7.5861 kHz. An analogue lowpass filter with the transfer function $H(s) = \frac{1}{s+1}$ has a 3dB corner frequency of 1 rad/sec. Using the lowpass to bandpass transformation $s' = \frac{s^2+\omega_l\omega_u}{s(\omega_u-\omega_l)}$, where ω_ll is the lower corner frequency and ω_u the upper corner frequency, together with the bilinear transform, design the required digital filter. Calculate the poles and zeros of the digital filter and show it gives the desired bandpass response.

9. A 4th order analogue Butterworth lowpass filter with cutoff frequency 1 rad/sec has poles at $s = -0.3827 \pm j0.9239$ and $-0.9239 \pm j0.3827$. Design a 4th order lowpass digital filter with sampling rate 8kHz, unit DC gain, and cutoff frequency 1kHz using the bilinear transform

$$s \rightarrow \frac{1-z^{-1}}{1+z^{-1}}$$

   Determine the coefficients of implementation using second order (biquadratic) section(s).

10. A digital filter has the transfer function $(1-bz^{-1})/(1-az^{-1})$, where $a > 0$, $b > 0$, $b < a$. Determine the scale factors needed for l_1 scaling, frequency response scaling, and l_2 scaling, assuming that the maximum magnitude

which can be represented in the filter internal arithmetic is the same as the maximum input magnitude. HINTS: for l_1 and l_2 scaling, use the standard expression for the sum of a Geometric Progression; for frequency-response scaling, note that you are only interested in the frequency at which the frequency response of interest is maximum - which is simple to determine in this case.

11. Consider the following system

$$y(n) = ay(n-1) - ax(n) + x(n-1).$$

(a) Show that it is all-pass, i.e. that $|H(\exp(j\Omega))| = 1$ for all Ω

(b) Obtain the direct form II realization of the system and sketch a diagram showing the signal delays, multiplications and additions from input to output.

(c) If you quantize the coefficients of the system in the direct form II, is it still all-pass?

(d) Obtain a realization by rewriting the difference equation as

$$y(n) = a[y(n-1) - x(n)] + x(n-1).$$

(e) If you quantize the coefficients of the system in this realization, is it still all-pass?

ANSWERS

1.

2. $S_0 = Nx_0$, $S_1 = Nx_{N-1}$, ..., $S_{N-2} = Nx_2$, $S_{N-1} = Nx_1$.

3.

4. $Y_k = (X_k + X^*_{N-k})/2$ and $Z_k = -j(X_k - X^*_{N-k})/2$

5. The 3 x 3 FFT requires 28 multiplications, and 36 additions. The 9-point DFT requires 64 complex mulltiplications, and 72 complex additions.

6. (a) delay = 1 sample; $|H(exp(j\Omega))| = 2 + 2cos(\Omega)$; lowpass;
   (b) delay = 1 sample; $|H(exp(j\Omega))| = 2 - 2cos(\Omega)$; highpass;
   (c) delay = 2 samples; $|H(exp(j\Omega))| = 2 - 2cos(2\Omega)$; bandpass;
   (d) delay = 1.5 samples; $|H(exp(j\Omega))| = 4cos(0.5\Omega) + 2cos(1.5\Omega)$; low-pass

7. (a) $\frac{1}{\pi n}\left(\sin\left(0.5\pi n\right) - \sin\left(0.4\pi n\right)\right)$
   (b)
   (c)

8. $H\left(z\right) = \frac{0.1}{1.4}\frac{1-z^{-2}}{1-z^{-1}+0.857z^{-2}}$;
   zeros at ± 1, poles at $0.9257e^{\pm j1.0}$; resonant frequency 7kHz

9. Section 1: FIR coeffs [1 2 1], feedback coeffs [-1.113, +0.5741]; section 2: FIR coeffs [1 2 1], feedback coeffs [-0.8554, +0.2097].

10.

11.

E. Punskaya 2010, S. Godsill 2011